Generalisations of graph broadcasts

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Contents

Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A dominating set S of a graph G is a subset of $V(G)$ such that each vertex in $V(G)$ is either in S itself or adjacent to a vertex in S. Domination and its variants have been well studied [11]. One variation introduced by Erwin in [9], involves studying a function $f : V(G) \to \{0, 1, 2, \ldots\}$ called a broadcast. We say a broadcast is dominating if for each vertex v there exists a vertex u with $f(u) \neq 0$ and $d_G(v, u) \leq f(u)$. The cost of a broadcast f is given by $\sum_{v \in V(G)} f(v)$ and we are usually interested in what the minimum cost is over all dominating broadcasts.

In a broadcast the cost to dominatate distance k is k. In this thesis we consider two models in which this need not be the case. The one model equips a graph with a cost function. This approach has been considered before in [14]. The other model equips the graph with a scaling function. We find a connection between the two frameworks, which links them in such a way that each framework proves results about the other.

Outline

Section 1

In Section 1 we introduce graphs and establish the notation used throughout this thesis. We then briefly introduce the notions of dominating and independent sets.

Section 2

In Section 2 we introduce Erwin's variant of domination [9], called broadcast domination. We define a broadcast on a graph and specify what is required for a broadcast to be dominating and minimal dominating. We define the cost of a broadcast and further define the broadcast domination number $\gamma_b(G)$ to be the smallest cost of any dominating broadcast. We find a number of bounds for the broadcast domination number, all of which are generalised in Section 5. The section ends by introducing the notion of an independent broadcast.

Section 3

In Section 3 we generalise broadcasts to allow for variable costs for broadcast distances. We first introduce the notion of a cost function. A cost function k is a nondecreasing function which associates, to each distance t we would like a vertex to broadcast, a cost $k(t)$. Thus we define the cost of a broadcast f on G with k to be $\sum_{v \in V(G)} (k \circ f)(v)$ and the broadcast domination number with respect to k to be the minimum cost over all dominating broadcasts on G with respect to k. We call this $\gamma_c^k(G)$. We then look at some examples and note that, by picking particular cost functions, we can emulate other forms of domination.

Next we introduce the scaling function which is a different way of going about associating a cost to a distance. Given a cost x, a scaling function g returns the maximum distance that can be broadcast for cost x. As one might expect, scaling functions are also nondecreasing. Instead of considering broadcasts on a graph G, we consider functions $h : V(G) \to \mathbb{N}$ called S-casts (short for scaled broadcast). Given an S-cast h and a scaling function g, we say it induces the broadcast $g \circ h$. An S-cast h is dominating if $g \circ h$ is a dominating broadcast. The cost of an S-cast h is given by $\sum_{v \in V(G)} h(v)$ and the S-cast domination number with respect to g is given by the smallest cost of a dominating broadcast. We call this $\gamma_s^g(G)$. We prove some minor results about S-casts and find a characterisation of minimal dominating S-casts.

We then make use of a basic concept in order theory, an adjunction between two nondecreasing functions. We start by describing a number of its basic properties before showing that technically and conceptually a cost function is the left adjoint of a scaling function and dually that a scaling function is the right adjoint of a cost function.

Finally we use the existence of this adjunction to show that if g is a scaling function and g^* is its left adjoint (a cost function) then $\gamma_s^g(G) = \gamma_c^{g^*}(G)$.

All the results regarding scaling functions are original.

Section 4

In Section 4 we introduce the notion of efficient S-casts and broadcasts and prove that if a scaling function g is superadditive then there exists an efficient optimal dominating S-cast. We then explore this result further and show that if g is not superadditive there exists a graph G where no efficient optimal dominating broadcasts exist. Finally we use the adjunction introduced in Section 3 to immediately deduce that if k is a subadditive cost function then there always exists an efficient optimal dominating broadcast.

The results about efficient optimal dominating S-casts are all original. The existence of efficient optimal dominating broadcasts for subadditive cost functions was already known [14], but the proof presented in this thesis is original.

Section 5

In Section 5 we generalise all of the bounds found in Section 2. We also prove other miscellaneous results about scaling functions and S-casts.

All the results in this section are original.

Section 6

In Section 6 we introduce the necessary background in complexity theory: complexity classes, Turing machines and oracles. These are the tools which will be used in analysing the complexity of the S-cast domination problem.

Section 7

In Section 7 we adapt the algorithm presented in [12] to prove that when g is a superadditive scaling function, the S-cast domination problem lies in $\mathcal{P}$ with respect to the oracle g.

Later we find a class of scaling functions for which the S-cast domination problem is $\mathcal{NP}$-hard.

The algorithm is due to Heggernes and Lokshtanov, but the adaptation of the algorithm which allows it to solve the S-cast domination problem is original. The result on $\mathcal{NP}$-hardness is original.

Section 8

In Section 8 we discuss bounds on S-cast domination number of product graphs. We generalise the results of [4] to the S-cast setting. The results generalise when we restrict the scaling functions to be linear and also when we restrict the graphs to trees and the scaling functions to be subadditive.

All the results in this section are original.

Section 9

In Section 9 we generalise the theory of scaling and cost functions to apply to more than just domination. We introduce something called a Γ-theory which essentially captures a broadcast property for which the relationship between cost and scaling functions is 'well behaved'. We show how this allows the results about the adjunction to be applied to the study of independent broadcasts and packing broadcasts.

Section 10

In Section 10 we summarise the major results appearing in this thesis.

1 Background

In this section we introduce graphs and related notation as well as dominating and independent sets of a graph, two notions which are generalised in later sections.

By $\mathbb{N}$ we denote the set $\{0, 1, 2, \ldots\}$ of non-negative integers. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the partially ordered set obtained from $\mathbb{N}$, where $\infty > n$ for any $n \in \mathbb{N}$. For a natural number k, we write $\underline{k}$ to mean $\{0, 1, \ldots, k\}$.

1.1 Graphs

A *graph* $G = (V, E)$ is an ordered pair comprising a set V of *vertices* and a set E of *edges*, where an edge is a two-element subset of V. For a graph $G = (V, E)$, we write $V(G)$ to refer to the vertex set of G and $E(G)$ to refer to the edge set. In this thesis we assume that V is finite. If $\{u, v\} \in E$ we say there is an edge connecting u and v and we often write $\{u, v\}$ as uv. Two vertices are said to be *adjacent* if there is an edge connecting the two. An edge e is said to be *incident* with a vertex v if $v \in e$. A walk W connecting two vertices v_1 and v_k is an alternating sequence of vertices and edges of the form $v_1, e_1, \ldots, v_i, e_i, \ldots e_k, v_{k+1}$ where e_i is incident with v_i and v_{i+1}. A walk from u to v is called a $u - v$ walk. A path $P = v_1, e_1, \ldots, v_i, e_i, \ldots e_k, v_{k+1}$ is a walk where all vertices are distinct. A path connecting u and v is a $u - v$ path. A cycle is a walk $v_i, e_i, \ldots, e_k, v_{k+1}$ of length $k \geq 3$ where each edge is distinct and $v_i = v_{k+1}$. The *length* of a walk W, denoted $|W|$, is the number of edges that appear in it. A *geodesic* between two vertices u and v is a $u - v$ path of minimum length. If u and v have a geodesic between them of length k, we say that the *distance* between u and v is k and we write $d_G(u, v) = k$. If there is no path connecting u and v we say $d(u, v) = \infty$. If x lies on a $u - v$ geodesic we say x lies between u and v. We call a graph *connected* if there is a path between any two vertices. The *eccentricity* of a vertex v, denoted $e(v)$, is the greatest distance from v to any vertex. The *diameter* of a graph G, denoted $\operatorname{diam}(G)$, is the largest eccentricity among all vertices while the *radius* of G denoted, $\operatorname{rad}(G)$, is the smallest. If $e(v) = \operatorname{rad}(G)$ we say v lies in the *centre* of G while if $e(v) = \operatorname{diam}(G)$ we say v lies in the *periphery* of G. We define the *open neighbourhood* of a vertex to be $N(v) = \{u \mid uv \in E\}$ and the *closed neighbourhood* $N[v] = N(v) \cup \{v\}$. The open neighbourhood of a set $S \subseteq V(G)$ is $N(S) = \bigcup_{v \in S} N(v) - S$ and the closed neighbourhood $N[S] = N(S) \cup S$. If $r \in \mathbb{N}$, then a *ball* of radius r centred on a vertex v we denote by $B(v, r) = \{w \in V(G) \mid d_G(v, w) \leq r\}$. Given a graph G and a set $S \subseteq V(G)$ we denote the induced *subgraph* generated by S by $G(S)$. The subgraph $G(S)$ has vertex set S and edge set $E(S) = \{e \in E \mid e \subseteq S\}$. A connected graph G with no cycles is called a tree. A digraph D is an ordered pair (V, A) of vertices and arcs with $A \subseteq V \times V - \{(v, v) \mid v \in V(G)\}$. Arcs are essentially directed edges between vertices. We say there is an arc from v to w if $(v, w) \in A$.

Unless otherwise stated, we assume that a graph G is nontrivial (i.e. has more than one vertex) and connected.

1.2 Domination and Independence

Definition 1.1. *A set $S \subseteq V(G)$ dominates a graph G if each $v \in V - S$ is adjacent to a vertex in S. If S dominates G we call S a dominating set. A dominating set S is minimal if no proper subset of S is dominating.*

If S is a dominating set of G and $u \in S$, we say u dominates each vertex adjacent to it as well as itself.

Example 1.2. *Let K_3 be the complete graph on three vertices with $V = \{u, v, w\}$ and $E = \{uv, uw, vw\}$. The set $S = \{u, w\}$ is dominating but not minimal. The set $T = \{u\}$ is minimal dominating (see Figure 1).*

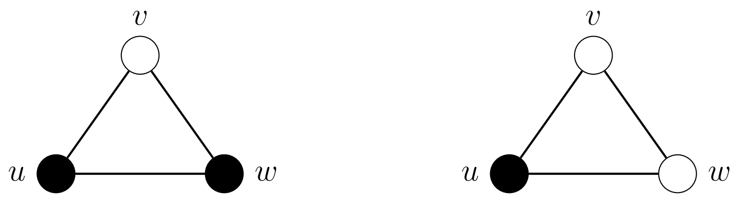

Figure 1: The black vertices belong to the dominating set. The dominating set on the graph on the left is not minimal, while the dominating set on the right is minimal dominating.

Definition 1.3. *Let G be a graph.*

1. *The domination number, $\gamma(G)$, is the cardinality of a smallest dominating set of G.*

2. *The upper domination number, $\Gamma(G)$, is the cardinality of a largest minimal dominating set of G.*

Definition 1.4. *Let G be a graph and $S \subseteq V(G)$. We say S is independent if no two vertices in S are adjacent. We call S maximal independent if no proper superset of S is independent.*

Definition 1.5. *Let G be a graph,*

1. *The independence number, $\beta(G)$, is the cardinality of a largest independent set of G.*

2. *The lower independence number, $i(G)$, is the cardinality of a smallest maximal independent set of G.*

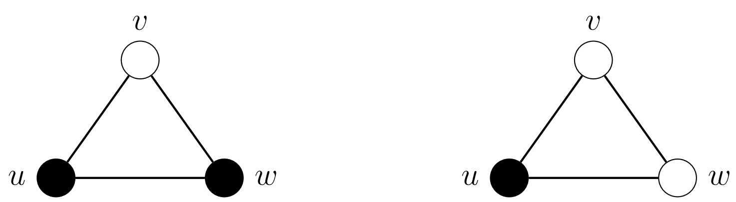

Figure 2: The dominating sets featured in Example 1.2. The one on the left is not independent while the one on the right is.

2 Broadcasts

There are a number of ways to generalise the notion of a dominating set. For instance, one can assume that each vertex in the dominating set dominates each vertex distance k away or less. This is called *distance-k domination* (introduced in [16]) and models a number of real world scenarios, like a museum trying to minimise the number of fire alarm sirens they need to install if each siren can be heard in rooms up to distance k away.

In reality there is no reason to expect that all sirens are equally loud and can be heard from the same number of rooms away. One might be quite soft and only audible up to two rooms away, while another may be louder and audible up to four rooms away. If we assume that for each positive integer k there exists a siren which can be heard up to k rooms away, then we can model this scenario with a function $f : V(G) \to \mathbb{N}$ which assigns to each vertex the distance k up to which it can be heard. Such a function f we call a broadcast and we can think of v as dominating each vertex within distance $f(v)$ of v when $f(v) \neq 0$.

Definition 2.1. *Given a graph G we call any function $f : V(G) \to \mathbb{N}$ a broadcast on G.*

This definition differs from the standard one as usually it is required that $f(v) \leq e(v)$ for each $v \in V(G)$. We find that this definition is more amenable to generalisation. (Removing this requirement does not affect the study of domination as that is concerned with finding minimal dominating broadcasts and a minimal dominating broadcast will always satisfy $f(v) \leq e(v)$ for each $v \in V(G)$.)

Example 2.2. *Let K_3 be the complete graph on three vertices with $V = \{u, v, w\}$ and $E = \{uv, uw, vw\}$. Then the function $f : V \to \mathbb{N}$ defined by $f(u) = 1 = f(w)$ and $f(v) = 0$, is a broadcast on K_3, as would be the function $h : V \to \mathbb{N}$ defined by $h(u) = 1$ and $h(w) = 0 = h(v)$ (see Figure 3).*

Given a broadcast f on a graph G we define the *broadcast neighbourhood* of a vertex v by $N_f[v] = \{u \in V(G) \mid d_G(u, v) \leq f(v)\}$. If $u \in N_f[v]$ we say that u is *f-dominated* by v. The *broadcast neighbourhood of a set S* is defined to be $N_f[S] = \bigcup_{v \in S} N_f[v]$. We define $V_f^0 = \{v \in V(G) \mid f(v) = 0\}$ and $V_f^+ = \{v \in V(G) \mid f(v) > 0\}$. The vertices in V_f^+ are called *broadcast vertices.* The *broadcast neighbourood of f* is denoted

Figure 3: In the broadcast f on the left we have $f(u) = 1 = f(w)$ while $f(v) = 0$. While in the broadcast on the right we have $h(u) = 1$ and $h(v) = 0 = h(w)$.

by $N_f[V_f^+] = \bigcup_{v \in V_f^+} N_f[v]$. Any vertex $u \in N_f[v]$ *hears* v's broadcast. Given a vertex $v \in V_f^+$ we define the *private f-neighbourhood* of v to be $\mathrm{pn}_f(v) = N_f[v] - \bigcup_{u \in V_f^+ - \{v\}} N_f[u]$. We define the *cost* of f to be $w(f) = \sum_{v \in V} f(v)$. We define $f_S : V(G) \to \mathbb{N}$ to be the *characteristic function* for the subset $S \subseteq V(G)$. That is $f_S(v) = 1$ if $v \in S$ and $f_S(v) = 0$ otherwise.

Given two functions f, g with domain A and codomain a poset $(B, \leq)$ we say $f \leq g$ if $f(a) \leq g(a)$ for every $a \in A$ and $f < g$ if $f \leq g$ and $f \neq g$.

Example 2.3. *Let $G_{4,2}$ be the graph with vertex set $V = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4)\}$ and edges connecting two vertices (a, b) and (c, d) if $(|a - c| = 1$ and $|b - d| = 0)$ or $(|a - c| = 0$ and $|b - d| = 1)$ (see Figure 4). Let $f : V \to \mathbb{N}$ with $f(1,1) = 2$, $f(2,2) = 1$ with $f(u) = 0$ for all other vertices u.*

The broadcast vertices are $V_f^+ = \{(1,1), (2,2)\}$, the broadcast neighbourhood of $(1,1)$ is $N_f[(1,1)] = \{(1,1), (1,2), (1,3), (2,1), (2,2)\}$ and of $(2,2)$ is $N_f[(2,2)] = \{(2,2), (2,1), (1,2), (2,3)\}$. The private f-neighbourhood of $(1,1)$ is $\mathrm{pn}_f(1,1) = \{(1,1), (1,3)\}$ and of $(2,2)$ is $\mathrm{pn}_f(2,2) = \{(2,3)\}$. The cost of f is $w(f) = f(1,1) + f(2,2) = 2 + 1 = 3$.

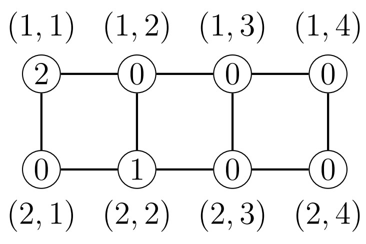

Figure 4: The broadcast $f : V(G_{4,2}) \to \mathbb{N}$ with $f(1,1) = 2, f(2,2) = 1$ and $f(v) = 0$ for $v \neq (1,1)$ or $(2,2)$, visualised. Each vertex in the graph is labelled with $f(v)$.

2.1 Dominating Broadcasts

As mentioned, if $v \in V_f^+$ we think of v as dominating all vertices distance $f(v)$ or less away. With this in mind we consider what it means to be a dominating broadcast.

Definition 2.4. *Let G be a graph.*

1. *A broadcast f on G is said to be dominating if for every vertex u there exists a vertex $v \in V_f^+$ such that $d_G(u, v) \leq f(v)$.*

2. *A dominating broadcast f is said to be minimal dominating if when f' is a broadcast on G such that $f' < f$, then f' is not dominating.*

Example 2.5. *Let $G_{4,2}$ be the graph with vertex and edge set as defined in Example 2.3. Then the function $f : V \to \mathbb{N}$ with $f(1,1) = 3$, $f(2,4) = 1$ and $f(u) = 0$ for all other vertices u, is dominating, but not minimal dominating, as the function $h : V \to \mathbb{N}$ with $h(1,1) = 2$, $h(2,4) = 1$ is dominating and indeed minimal (see Figure 5).*

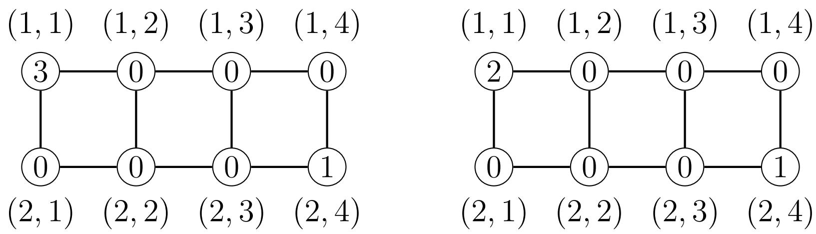

Figure 5: The figure on the left represents the broadcast f and the figure on the right the broadcast h (both as described in Example 2.5)

We now define the broadcast versions of the domination numbers.

Definition 2.6. *For a graph G, the broadcast domination number is*

$$\gamma_b(G) = \min\{\sum_{v \in V} f(v) \mid f \text{ is a dominating broadcast on } G\}.$$

The upper broadcast domination number is

$$\Gamma_b(G) = \max\{\sum_{v \in V} f(v) \mid f \text{ is a minimal dominating broadcast on } G\}$$

We call f an optimal dominating broadcast if $\sum_{v\in V} f(v) = \gamma_b(G)$.

We now prove some results on broadcasts and remind the reader that all graphs G are assumed to be connected and nontrivial.

Proposition 2.7. *For a graph G we have the inequality.*

$$\gamma_b(G) \leq \min\{\gamma(G), \mathrm{rad}(G)\} \leq \max\{\Gamma(G), \mathrm{diam}(G)\} \leq \Gamma_b(G)$$

Proof. Let G be a graph and let $v \in V(G)$ lie in its centre. Then the function $f : V(G) \to \mathbb{N}$ with $f(v) = \mathrm{rad}(G)$ and $f(u) = 0$ when $u \neq v$ is dominating.

Let S be a dominating set on G with cardinality $\gamma(G)$. Then f_S is a dominating broadcast. Since $w(f) = \mathrm{rad}(G)$ and $w(f_S) = \gamma(G)$, it follows that $\gamma_b(G) \leq min\{\mathrm{rad}(G), \gamma(G)\}$.

Two similar constructions give that
$max\{\mathrm{diam}(G), \Gamma(G)\} \leq \Gamma_b(G)$. □

A useful characterisation of minimal dominating broadcasts is given below.

Theorem 2.8. *(Erwin [9]) Let f be a dominating broadcast on a graph G. Then f is minimal if and only if the following two conditions are satisfied.*

1. *Every vertex v with $f(v) \geq 2$ has a private f-neighbour that is at distance $f(v)$ from v.*
2. *Every vertex v with $f(v) = 1$ has a private f-neighbour in $N[v]$.*

Proof. Let f be a minimal dominating broadcast on G. Consider a vertex $v \in V(G)$ with $f(v) \geq 2$ and assume it has no private f-neighbour distance $f(v)$ away. Then the function f' with $f'(u) = f(u)$ for $u \neq v$ and $f'(v) = f(v) - 1$ is dominating, contradicting the minimality of f.

Now let v be a vertex with $f(v) = 1$ and suppose v has no private f-neighbours. Then all the other broadcast vertices f-dominate the graph and so f is not minimal, which is again a contradiction. Hence when f is minimal, the above two conditions have to hold.

Next consider a dominating broadcast f on G where conditions (1) and (2) hold. Consider a vertex $v \in V(G)$ with $f(v) \geq 2$. It has a private f-neighbour w distance $f(v)$ away. Any function f' with $f'(u) = f(u)$ for $u \neq v$ and $f'(v) < f(v)$ is not dominating as w is not dominated.

In a similar way if we have a vertex v with $f(v) = 1$ then this vertex can not be removed from the broadcast vertices without leaving its private f-neighbour not f-dominated. Hence f must be a minimal dominating broadcast. $\square$

We want to prove a bound relating the broadcast domination number to the number of edges in the graph G. Before we can prove it we require two lemmas.

Lemma 2.9. *(Erwin [9]) Let f be a dominating broadcast on G and take $u, v \in V_f^+$ with $u \neq v$ and let u_p, v_p be private f-neighbours of u and v respectively. For any two vertices x and y in G, if x lies on a geodesic between u and u_p and y lies on geodesic between v and v_p, then $x \neq y$.*

Proof. Consider $u, v \in V_f^+$ and let u_p and v_p be private neighbours of u and v respectively. Assume that a vertex w lies on a geodesic between u and u_p and on a geodesic between v and v_p. Without loss of generality let $f(u) - d_G(u, w) \geq f(v) - d_G(v, w)$. We know that all vertices x with $d_G(w, x) \leq f(u) - d_G(u, w)$ are f-dominated by u. Furthermore we know that $d_G(v_p, w) \leq f(v) - d_G(v, w) \leq f(u) - d_G(u, w)$. Hence it follows that v_p is dominated by u, which is a contradiction. $\square$

Lemma 2.10. *(Erwin [9]) Let G be a graph and f a minimal dominating broadcast on G. Then for each $v \in V_f^+$, there exists an edge incident with v not incident with any other $u \in V_f^+$.*

Proof. Suppose that this is not the case. Then there exists a vertex v such that each edge incident with v is incident with another vertex in V_f^+. Let v_p be a private f-neighbour of v and consider a $v - v_p$ geodesic. We know each neighbour of v is in V_f^+ hence $d(v, v_p) \geq 2$. We know that there must lie another vertex $u \in V_f^+$ on this geodesic. If $f(u) \geq f(v) - 1$ then v_p is not a private f-neighbour of v, so $f(u) < f(v) - 1$. But then every private f-neighbour of u is at distance at most $f(v) - 1$ from v. But then v f-dominates u_p which is a contradiction. $\square$

Theorem 2.11. *([8]) Let G be a graph with m edges. Then $\Gamma_b(G) \leq m$.*

Proof. Let f be a minimal dominating broadcast. For every vertex v with $f(v) \geq 2$ let v_p be a private f-neighbour of v distance $f(v)$ from v (which we know exists from Theorem 2.8), let P_v be a $v - v_p$ geodesic and let $E_v = E(P_v)$ be the set of all edges in P_v. For vertices v with $f(v) = 1$, let E_v be a set containing a single edge incident to v but not incident to any other $u \in V_f^+$ (which we know exists by Lemma 2.10).

Certainly $|E_v| = f(v)$ for each $v \in V_f^+$. To show for $v, u \in V_f^+$ that E_v and E_u are pairwise disjoint consider first $A = \{v \in V(G) \mid v \in V_f^+ \text{ and } f(v) \geq 2\}$. By Lemma 2.9 we know that $E_v \cap E_u = \emptyset$ for $v, u \in A$. Next let $B = V_f^+ - A$. Trivially $E_v \cap E_u = \emptyset$ for $u, v \in B$, so all that needs to be checked is that $E_v \cap E_u = \emptyset$ for $v \in A$ and $u \in B$.

Let $v \in A$ and $u \in B$ and assume $E_v \cap E_u \neq \emptyset$. It follows then that u lies on P_v. Clearly $u \neq v_p$ and hence $d_G(v, u) \leq f(v) - 1$. But then every vertex in $N[u]$ is f-dominated by v, which is a contradiction. Hence it follows that the E_v's are pairwise disjoint.

Thus it follows that $\Gamma_b(G) \leq m$.

□

2.2 Independent Broadcasts

Independence can be generalised to broadcasts by requiring that no broadcast vertex is able to hear any other broadcast vertex.

Definition 2.12. *Let G be a graph.*

1. *A broadcast f is independent if $N_f[v] \cap V_f^+ = \{v\}$, for each $v \in V_f^+$.*
2. *A broadcast f is maximal independent if it is independent and has the property that any broadcast f' with $f < f'$ is not an independent broadcast.*

Definition 2.13. *For a graph G, the broadcast independence number is*

$$\beta_b(G) = \max\{\sum_{v \in V} f(v) \mid f \text{ is an independent broadcast}\}.$$

The lower broadcast independence number is

$$i_b(G) = \min\{\sum_{v \in V} f(v) \mid f \textit{ is a maximal independent broadcast}\}.$$

Example 2.14. *Let G be the graph shown in Figure 10 with vertex set $V = \{(1,3), (2,3), (3,1), (3,2), (3,3), (3,4), (3,5), (4,3), (5,3)\}$ and edges connecting two vertices (a,b) and (c,d) if $(|a-c| = 1$ and $|b-d| = 0)$ or $(|a-c| = 0$ and $|b-d| = 1)$.*

Let $f : V \to \mathbb{N}$ with $f(1,3) = f(3,1) = f(3,5) = f(5,3) = 3$ and $f(u) = 0$ for all other vertices u. Then f is a maximal independent broadcast.

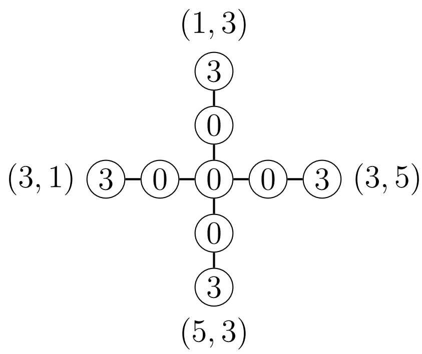

Figure 6: A maximal independent broadcast.

3 Generalised Broadcasts

Consider again the example of placing fire alarms in a museum. Two assumptions are made when modelling this scenario with broadcasts: the cost of an alarm which broadcasts up to distance k, is itself k and that for every positive integer t there exists a fire alarm which broadcasts precisely up to distance t.

We now introduce two more general frameworks, in which these assumptions need not be true. The first equips a graph with a function $k : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ called the cost function, which assigns to every distance t the cost for a vertex to dominate that distance. This allows us to do away with the first assumption.

The second model does away with both assumptions and here we equip a graph with a function $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$, called a scaling function, which assigns to each natural number n the distance a vertex can broadcast for cost n.

3.1 Cost Functions

Consider a model in which the cost of dominating a vertex is determined by how far away it is. Such an approach has been considered in [12] and in [14], where it is proved that a minimal cost dominating broadcast can be found in polynomial time when the cost function is subadditive (see Section 7).

We shall assume that the domain and codomain of a cost function are both $\overline{\mathbb{N}}$.

Definition 3.1. *A function $k : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ is a cost function if the following conditions hold:*

1. $k(0) = 0$,
2. *k is nondecreasing,*
3. $k(\infty) = sup\{k(x) \mid x \in \mathbb{N}\}$.

The function k has $k(0) = 0$ because it should cost nothing for a vertex to broadcast no distance. It is nondecreasing because we would expect it to cost more to broadcast further.

The reason for the inclusion of the condition that $k(\infty) = \sup\{k(x) \mid x \in \mathbb{N}\}$ will become clear in Section 3.4.

Let G be a graph and k a cost function. A broadcast on (G, k) is a function $f : V(G) \to \mathbb{N}$. A broadcast f is *dominating* on (G, k) if f is dominating on G, furthermore we say f is *minimal dominating* on (G, k) if and only if f is minimal dominating on G.

The *cost* of a broadcast f on (G, k) is given by

$$w_c^k(f) = \sum_{v \in V} (k \circ f)(v).$$

Example 3.2. *Let $G_{4,2}$ be the graph defined in Example 2.3 and $k : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$, the cost function given by $k(x) = 2x$ for $x \in \mathbb{N}$ and $k(\infty) = \infty$. Let h be the minimal dominating broadcast defined in Example 2.5, which has $h(1,1) = 2$ and $h(2,4) = 1$. This broadcast has cost equal to $w_c^k(h) = \sum_{v \in V}(k \circ h)(v) = 6$.*

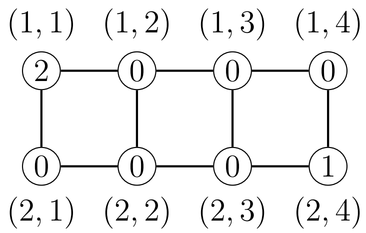

Figure 7: The broadcast h visualised on $(G_{4,2}, k)$ as described in example 3.2.

Below we define some useful parameters.

Definition 3.3. *For a graph G and a cost function k, the cost domination number of (G, k) is*

$$\gamma_c^k(G) = \min\{w_c^k(f) \mid f \textit{ is a dominating broadcast on } (G, k)\}.$$

The upper cost domination number with respect to k is

$$\Gamma_c^k(G) = \max\{w_c^k(f) \mid f \textit{ is a minimal dominating broadcast on } (G, k)\}.$$

We call dominating broadcasts f with $w_c^k(f) = \gamma_c^k(G)$ optimal dominating. With respect to a cost function an optimal dominating broadcast can fail to be minimal.

Example 3.4. *Let $k : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ be the cost function with $k(0) = 0, k(1) = 1 = k(2)$ and $k(n) = \infty$ for all $n \in \overline{\mathbb{N}}$ with $n \geq 3$. Then let P_2 be the graph with $V(G) = \{u, v\}$ and $E(G) = \{uv\}$. The broadcast f with $f(u) = 2$ and $f(v) = 0$ is optimal, but not minimal.*

Broadcast domination with cost functions generalises both classical domination and broadcast domination.

Example 3.5. *Let G be a graph and k the cost function with $k(1) = 1$ and $k(n) = \infty$ for $n \geq 2$.*

Any dominating broadcast f with finite cost on (G, k) is a characteristic function and consequently corresponds to a dominating set. Furthermore every dominating set on G corresponds to a broadcast on (G, k). Thus studying the broadcasts on (G, k) is the same as studying the dominating sets on G. In addition the cost of each broadcast f on (G, k) is equal to the size of the dominating set f corresponds to. Thus the domination numbers coincide.

Example 3.6. *Let G be a graph and k a cost function with $k(n) = n$.*

Every broadcast f on (G, k) is also a broadcast on G and vice versa. Furthermore since k is the identity function the cost of f on (G, k) is the same as the cost of f on G.

Distance-k domination can be partially generalised.

Example 3.7. *Let G be a graph and t a cost function with $t(x) = 1$ for $0 < x \leq k$ and $t(x) = \infty$ for $x > k$.*

Every k-dominating set S on G corresponds to a broadcast f_S^k where $f_S^k(v) = k$ when $v \in S$ and $f_S^k(u) = 0$ when $u \notin S$. We get $w_c^t(f_S) = |S|$. Also if we let f be a $\gamma_c^t(G)$-dominating broadcast then we have $w_c^t(f) = |V_f^+|$. Letting $S = V_f^+$ we have that S is k-dominating as f is dominating and $f(v) \leq k$ for each $v \in V_f^+$. Thus we conclude that $\gamma^k(G) = \gamma_c^t(G)$.

Studying broadcasts on (G, t) is not the same as studying k-dominating sets, as broadcast with $0 < f(v) < k$ do not naturally correspond to any k-dominating sets.

3.2 S-casts

In the previous section, we considered a generalisation of the idea of a broadcast in which the cost of dominating at a distance t is not necessarily equal to t. We now consider a second generalisation in which a graph is equipped with a function $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ which assigns to each $n \in \overline{\mathbb{N}}$ the distance a vertex dominates for cost n. We call g a scaling function. If $g(n) = d$ we say paying n buys distance d. We formalise this with the following definition.

Definition 3.8. *Given a graph G, we call a function $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ a scaling function if the following conditions hold:*

1. $g(0) = 0$,
2. *g is nondecreasing,*
3. $g(\infty) = \infty$.

It makes sense that $g(0) = 0$ since paying 0 should buy 0 distance. Also g is nondecreasing because paying more should never buy less distance.

The reason for the inclusion of the condition that $g(\infty) = \infty$ will become clear in Section 3.4.

Example 3.9. *The function $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ with $g(x) = \lfloor \frac{1}{2}x \rfloor$ for $x \in \mathbb{N}$ and $g(\infty) = \infty$, is a scaling function.*

Definition 3.10. *Given a graph G and a scaling function g, an S-cast (short for scaled broadcast) on (G, g) is a function $h : V(G) \to \overline{\mathbb{N}}$.*

Let G be a graph and g be a scaling function. Let h be an S-cast on (G, g). We call $g \circ h$ the *broadcast induced by h*. The S-cast vertices V_h^+ are the vertices with $h(v) \neq 0$. We define the *S-cast neighbourhood of a vertex v* to be $N_h[v] = \{u \in V(G) \mid d_G(u, v) \leq (g \circ h)(v)\}$. If $u \in N_h[v]$ we say u is *h-dominated* by v. The *S-cast neighbourhood of a set S* is defined to be $N_h[S] = \bigcup_{v \in S} N[v]$. If $u \in N_h[S]$ we say u is h-dominated by S. The S-cast neighbourhood of h is $N_h[V_h^+]$. The *private h-neighbours* of an S-cast vertex v is given by

$\mathrm{pn}_h(v) = N_h[v] - \bigcup_{u \in V_h^+ - \{v\}} N_h[u]$. The *cost* of an S-cast h is $w_s^g(h) = \sum_{v \in V} h(v)$. This is in some sense dual to the model involving cost functions where the broadcast is the function f but where you must compose it with the cost function k to find its cost.

Example 3.11. *Let $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ be a scaling function with $g(0) = 0$, $g(x) = x + 1$ for $x \in \mathbb{N} - \{0\}$ and $g(\infty) = \infty$. Then for any S-cast h on (G, g) and $v \in V(G)$, we have $g \circ h(v) \neq 1$.*

Example 3.11 shows that not all broadcasts on G can be induced by S-casts on (G, g).

Below we consider the relationship between broadcasts on G and S-casts on (G, g).

Proposition 3.12. *Consider a graph G, scaling function g and broadcast f on G. There exists an S-cast h on (G, g) such that $f = g \circ h$ if and only if $Im_f(V(G)) \subseteq Im_g(\overline{\mathbb{N}})$.*

Proof. Suppose that $\mathrm{Im}_f(V) \subseteq \mathrm{Im}_g(\overline{\mathbb{N}})$. For each $v \in V(G)$, we know that there exists an $x \in \overline{\mathbb{N}}$ such that $f(v) = g(x)$. Choose h such that $h(v) = x$ and we get that $g(h(v)) = f(v)$ as desired.

For the other direction, if we know that there exists an S-cast h such that $g \circ h = f$ we clearly have that $\mathrm{Im}_f(V(G)) \subseteq \mathrm{Im}_g(\mathbb{N})$. $\square$

Definition 3.13. *A scaling function g is called practically surjective on G if for each $x \in \underline{\mathrm{diam}(G)}$ there exists $n \in \mathbb{N}$ with $g(n) = x$.*

Corollary 3.14. *If g is a practically surjective scaling function then for any broadcast f we can find a S-cast h such that $g \circ h = f$.*

Finally note that if $f = g \circ h$ and $f' = g \circ h'$ are broadcasts and $f' < f$, then it need not be the case that $h' \leq h$.

Example 3.15. *Let G be a graph and $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ any scaling function with $g(1) = 1$ and $g(2) = 1$. Let $v_1, v_2 \in V(G)$ with $v_1 \neq v_2$. Suppose h is any S-cast with $h(v_1) = 1$, $h(v_2) = 1$ and h' is an S-cast with $h'(v_1) = 2$, $h'(v_2) = 0$ and $h'(u) = h(u)$ for all $u \in V(G)$ with $u \neq v_1, v_2$. Then $g \circ h' < g \circ h$ but $h' \not\leq h$.*

3.3 Dominating S-casts

In this section we define and observe some basic properties of dominating S-casts.

Definition 3.16. *Let G be a graph and g a scaling function.*

1. *An S-cast h is dominating if $g \circ h$ is a dominating broadcast.*
2. *An S-cast h is minimal dominating if h is dominating and any S-cast k with $k < h$, is not dominating.*

Example 3.17. *Let $G_{4,2}$ be the graph defined in Example 2.3 and $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$, the scaling function given by $g(x) = 2x$ for $x \in \mathbb{N}$ and $g(\infty) = \infty$. Consider an S-cast h on (G, g), with $h(1,1) = 1$ and $h(2,4) = 1$. Since $\operatorname{rad}(G) > 2$, there is no dominating S-cast on (G, g) with cost 1. Thus h is a minimal dominating S-cast. Note that $g \circ h$ is not a minimal dominating broadcast as the broadcast f given by $f(1,1) = 2$, $f(2,4) = 1 < (g \circ h)(2,4)$ and $f(v) = 0$ for all other v, is dominating.*

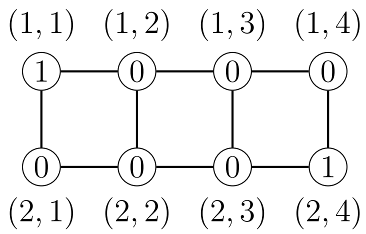

Figure 8: The S-cast h defined on (G, g) as described in Example 3.17.

Below we define the domination numbers.

Definition 3.18. *For a graph G and a scaling function g, the S-cast domination number with respect to g is*

$$\gamma_s^g(G) = \min\{\sum_{v \in V(G)} h(v) \mid g \circ h \text{ is a dominating broadcast}\}.$$

The upper S-cast domination number with respect to g is

$$\Gamma_s^g(G) = \max\{\sum_{v \in V(G)} h(v) \mid h \text{ is a minimal dominating S-cast}\}.$$

Just as with cost functions, other types of domination can be generalised by choosing the right scaling functions. With scaling functions, standard domination and distance-k domination are generalised if we assume that only minimal dominating S-casts are the objects of study. Broadcast domination is completely generalised.

Example 3.19. *Let G be a graph. To generalise broadcast domination consider the scaling function $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ such that $g(n) = n$ for all $n \in \mathbb{N}$ and $g(\infty) = \infty$.*

For every S-cast h we have $g \circ h = h$. Consequently studying S-casts on (G, g) is the same as studying broadcasts on G.

Example 3.20. *Let G be a graph. To generalise standard domination consider the scaling function $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ $g(n) = 1$ for all $n \in \mathbb{N}$ and $g(\infty) = \infty$.*

If h is a minimal dominating S-cast on (G, g), with $h(v) = 1$ for each $v \in V_f^+$, h is a characteristic function and corresponds to a dominating set X on G. Furthermore if X is a minimal dominating set on G, then f_X is a minimal dominating S-cast on (G, g). Also since h is minimal dominating $w_s^g(h) = |V_h^+|$ and so we conclude that $\gamma(G) = \gamma_s^g(G)$.

Example 3.21. *Let G be a graph. To generalise distance-k domination consider the scaling function $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ such that $g(0) = 0$, $g(n) = k$ for all $n \in \mathbb{N}$ and $g(\infty) = \infty$.*

All minimal dominating S-casts on (G, g) are characteristic functions and using a similar argument as in Example 3.20 we see that distance-k domination has been generalised

The next example shows that for a scaling function g and an S-cast h, if $g \circ h$ is a minimal dominating broadcast then it need not be the case that h is a minimal dominating S-cast.

Example 3.22. *Let G be a graph and g the scaling function given by $g(0) = 0, g(n) = 1$ for all $n \in \mathbb{N}$. Let X be a dominating set with cardinality equal to $\gamma(G)$. Then h_X is a minimal dominating S-cast and $g \circ h_S$ is a minimal dominating broadcast. If we define h to be a broadcast with $h(v) = 2$ when $v \in V_{h_X}^+$ and $h(u) = 0$ otherwise then $g \circ h$ is also a minimal dominating broadcast but h is not a minimal dominating S-cast.*

Proposition 3.23. *Let G be a graph and g a practically surjective scaling function. If h is an S-cast with $w_s^g(h) = \gamma_s^g(G)$ then $g \circ h$ is a minimal dominating broadcast.*

Proof. Assume to the contrary that $g \circ h$ is not minimal. Then there exists a vertex $v \in V(G)$ and a dominating broadcast f such that $f(v) < (g \circ h)(v)$ but with $f(u) = (g \circ h)(u)$ for all $u \neq v$. Because g is practically surjective there exists an n such that $g(n) = f(v)$. Furthermore, because g is nondecreasing we know that $n < h(v)$. Thus the function h', which we define to be identical to h except that $h'(v) = n$, is an S-cast since $f = g \circ h'$. It is clear from the construction of h' that $\sum_{v \in V} h'(v) < \sum_{v \in V} h(v)$, which is a contradiction. $\square$

Finally we prove a simple but very useful result.

Lemma 3.24. *If h is a minimal dominating S-cast on a graph G with scaling function g then $V_h^+ = V_{g \circ h}^+$.*

Proof. If $v \in V_h^+$ then $g \circ h(v) \neq 0$ or else h is not minimal. Furthermore if $(g \circ h)(v) \neq 0$ then $h(v) \neq 0$. $\square$

We characterised minimal dominating broadcasts earlier which we now generalise to the S-cast case.

Proposition 3.25. *Let G be a graph, g a scaling function and h a dominating S-cast on (G, g). Then h is a minimal dominating S-cast if and only if the following two conditions are satisfied.*

1. *Every vertex v with $(g \circ h)(v) \geq 2$ has a private $(g \circ h)$-neighbour that is a distance greater than $g(h(v) - 1)$ from v.*
2. *Every vertex v with $(g \circ h)(v) = 1$ has a private $(g \circ h)$-neighbour in $N[v]$ and $g(h(v) - 1) = 0$.*

Proof. Let h be a minimal dominating S-cast. Consider a vertex $v \in V(G)$ with $(g \circ h)(v) \geq 2$ and assume it has no private $(g \circ h)$-neighbour further than distance $g(h(v) - 1)$ away. Then the S-cast h' with $h'(u) = h(u)$ for $u \neq v$ and $h'(v) = h(v) - 1$ is dominating, contradicting the minimality of h.

If we consider a vertex v with $(g \circ h)(v) = 1$ and no private-h-neighbours then all the other broadcast vertices already together h-dominate the graph and so h is not minimal, which is a contradiction. Furthermore if $g(h(v) - 1) \neq 0$ then $g(h(v) - 1) = 1$. Then the S-cast h' given by $h'(u) = h(u)$ for $u \neq v$ and $h'(v) = h(v) - 1$ is dominating, contradicting the minimality of h. Hence when h is minimal dominating, the above two conditions have to hold.

Next consider a dominating S-cast h on (G, g) where the above conditions hold.

Consider a vertex $v \in V(G)$ with $(g \circ h)(v) \geq 2$. It has a private $(g \circ h)$-neighbour w distance greater than $g(h(v) - 1)$ away. A function h' with $h'(u) = h(u)$ for $u \neq v$ and $h'(v) < h(v)$ is not dominating as w does not lie in any $N_{g \circ h'}[u]$ for any $u \in V(G)$.

In a similar way, if we have a vertex v with $(g \circ h)(v) = 1$ then any function h' with $h'(u) = h(u)$ for $u \neq v$ and $h'(v) < h(v)$ has $(g \circ h')(v) = 0$, leaving the private neighbour of v undominated. Hence h must be minimal dominating. □

3.4 Adjoints

In this section we investigate the relationship between cost and scaling functions. We show that they form a Galois Connection (defined below). We show that every scaling function has a left adjoint and that this adjoint is a cost function. Furthermore each cost function is shown to have a right adjoint and that this right adjoint is a scaling function.

The theorems are borrowed from order theory and so below we introduce some terminology from the field. A partially ordered set or *poset* is a set equipped with a binary relation $\leq$ which is reflexive ($a \leq a$), anti-symmetric ($a \leq b$ and $b \leq a$ implies $a = b$) and transitive ($a \leq b$ and $b \leq c$ implies $a \leq c$).

Given a subset S of a partially ordered set we can consider its *meet* $\bigwedge S$ or *join* $\bigvee S$ which is respectively the infimum or suprememum of the set. Neither of these need exist. The empty join is the smallest element larger than every element in the empty set and so is the minimum element if it exists. Dually the empty meet is the maximum element if it exists.

We only consider nondecreasing functions between posets i.e. functions f where if $a \leq b$ then $f(a) \leq f(b)$. We say that a nondecreasing function f *preserves meets* if $f(\bigwedge S) =$

$\bigwedge f(S)$ and *preserves joins* if $f(\bigvee S) = \bigvee f(S)$.

For more on this topic see [7].

Definition 3.26. *Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. A monotone Galois connection between these posets consists of two nondecreasing functions: $f : B \to A$ and $g : A \to B$, such that for all a in A and b in B, we have $f(b) \leq a$ if and only if $b \leq g(a)$.*

We call f the *left adjoint* of g and from here on denote it by g^*. We call g the *right adjoint* of f and call it f_*. This can be done because right and left adjoints are unique which is shown in the next result. Another name for a Galois connection is an *adjunction.*

Example 3.27. *If f is a bijection then f^{-1} is both its right and left adjoint.*

Example 3.28. *Let $g : \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3\}$ with both sets equipped with the standard ordering. Let*

$$g(0) = 0,$$

$$g(1) = 1,$$

$$g(2) = 1,$$

$$g(3) = 2,$$

$$g(4) = 3.$$

Then the left adjoint $g^* : \{0, 1, 2, 3\} \to \{0, 1, 2, 3, 4\}$ of g is given by

$$g^*(0) = 0,$$

$$g^*(1) = 1,$$

$$g^*(2) = 3,$$

$$g^*(3) = 4.$$

Proposition 3.29. *Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets and let $g : (A, \leq) \to (B, \leq)$ be a nondecreasing function. Then a function g' is left adjoint to g if and only if $g'(b)$ is the least element a such that $b \leq g(a)$.*

Proof. Assume g and g' are adjoint. If $g'(b) = a$ then from the definition $b \leq g(a)$. If a is not the least element, then there exists a c such that $a \not\leq c$ with $b \leq g(c)$, which implies that $g'(b) \leq c$. This is in conflict with out assumption that $g'(b) = a \not\leq c$.

Now assume that $g'(b)$ is the least element a such that $b \leq g(a)$ for each $b \in B$. Assume that $g'(b) \leq c$ then $b \leq g(g'(b)) \leq g(c)$. Next assume that $b \leq g(c)$. We know that $b \leq g(g'(b))$, but since $g'(b)$ is the least element for which this is the case, $g'(b) \leq c$. $\square$

The following dual result can be shown similarly.

Proposition 3.30. *Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets and let $g : (A, \leq) \to (B, \leq)$ be a nondecreasing function, then $g_*(b)$ is the largest element a such that $g(a) \leq b$.*

These results give that the right and left adjoints of a given function are uniquely defined.

Galois connections give rise to closure and interior operators defined below.

Definition 3.31. *A closure operator $cl : X \to X$ on a poset X is a function satisfying the following three axioms.*

- *$x \leq cl(x)$ (extensive)*
- *If $x \leq y$ then $cl(x) \leq cl(y)$ (nondecreasing)*
- *$cl(cl(x)) = cl(x)$ (idempotent).*

If we swap the order of the extensive axiom around ($cl(x) \leq x$) and keep the remaining two axioms then we call cl an interior operator.

If $cl : X \to X$ is a closure or interior operator then we call the elements $x \in X$ with $cl(x) = x$ the closed elements. Also we will commonly write $\overline{x}$ to refer to the closure of x.

Proposition 3.32. *Suppose $g : (A, \leq) \to (B, \leq)$ and $g^* : (B, \leq) \to (A, \leq)$ form a Galois connection. Then $g \circ g^* : (B, \leq) \to (B, \leq)$ is a closure operator and $g^* \circ g : (A, \leq) \to (A, \leq)$ is an interior operator.*

Proof. Consider $g \circ g^*$. We have that $g^*(x)$ is the smallest b such that $g(b) \geq x$. Consequently $x \leq g(g^*(x))$. A similar argument show that $g^*(g(x)) \leq x$.

Next let $x \leq y$. We get the inequality chain $x \leq y \leq g(g^*(y))$. By the definition of a Galois connection we get $g^*(x) \leq g^*(y)$ and then using the fact that g is nondecreasing gives $g(g^*(x)) \leq g(g^*(y))$. This exact argument gives that $g^* \circ g$ is nondecreasing too.

In order to prove idempotency we will first show that $g^* \circ g \circ g^* = g^*$. We have already shown that $g \circ g^*$ is extensive and so $x \leq g(g^*(x))$. Also since g^* is nondecreasing we get $g^*(x) \leq g^*(g(g^*(x)))$. Again since g^* is nondecreasing and $g(g^*(x)) \leq x$, we get $g^*(g(g^*(x))) \leq g^*(x)$. Thus $g^*(g(g^*(x))) = g^*(x)$. Taking this equality and composing with g on the left gives $(g \circ g^*) \circ (g \circ g^*) = g \circ g^*$. And composing with g on the right gives gives $(g^* \circ g) \circ (g^* \circ g) = g^* \circ g$. This completes the proof. □

Below is a special case of Freyd's Adjoint Functor Theorem applied to posets.

Theorem 3.33. *If $g : (A, \leq) \to (B, \leq)$ is any nondecreasing map such that $(A, \leq)$ has all meets and g preserves all meets, then g has a left adjoint.*

Proof. We construct the adjoint and prove it has the required properties. Let $g'(b) = \inf\{a \in A \mid b \leq g(a)\}$ and consider

$$\begin{aligned} g(g'(b)) &= g(\inf\{a \in A \mid b \leq g(a)\}) \\ &= \inf\{g(a) \mid b \leq g(a)\}. \end{aligned}$$

It follows that $b \leq g(g'(b))$ and that in fact it is the least element for which this is the case. Thus the function described is the left adjoint of g by Proposition 3.29. □

It can be similarly shown that the dual theorem below is true.

Theorem 3.34. *If $k : (B, \leq) \to (A, \leq)$ is any nondecreasing map such that $(B, \leq)$ has all joins and k preserves all joins, then g has a right adjoint.*

Recall that taking the meet of a set of elements means to find the infimum and to do the same but for joins means to take the supremum.

Theorem 3.35. *If $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ is a scaling function, then g has a left adjoint and this left adjoint is a cost function.*

Proof. First we check that $\overline{\mathbb{N}}$ has all meets. By the well ordering property of the naturals every nonempty set has a least element. The addition of the point ∞ does not change this. Furthermore the empty meet is $\infty \in \overline{\mathbb{N}}$ and so it has all meets.

It remains to check that g preserves meets. By the well ordering principle, when A is nonempty $\inf(A) = \min(A)$. Thus $g(\inf(A)) = g(\min(A)) = \min(g(A)) = \inf(g(A))$ because g is nondecreasing. If A is empty then $\inf(A) = \infty$ and since g is a scaling function $g(\inf(A)) = g(\infty) = \infty$.

Now we show that g^* is a cost function. Certainly g^* is nondecreasing and $g^*(0) = 0$ so it remains to verify that $g^*(\infty) = \sup\{g^*(x) \mid x \in \mathbb{N}\}$. We split this into two cases. First assume $g^*(\infty) = k \in \mathbb{N}$. Then $g(k) = \infty$ and $g(k-1) = t \in \mathbb{N}$. Thus g returns no finite numbers greater than t and so $g(t') = k$ for each $t' > t$ and so $\sup\{g^*(x) \mid x \in \mathbb{N}\} = k = g(\infty)$. Next assume $g^*(\infty) = \infty$. Then $g(a) = \infty$ only when $a = \infty$. So either g is unbounded in which cases g^* is unbounded and $\sup\{g^*(x) \mid x \in \mathbb{N}\} = \infty$, otherwise g is eventually k for $k \in \mathbb{N}$ in which case $g^*(k') = \infty$ for each $k' > k$ and again $\sup\{g^*(x) \mid x \in \mathbb{N}\} = \infty$. This completes the proof. □

Example 3.36. *Let g be the scaling function which gives classical domination, namely $g(0) = 0, g(n) = 1$ for all $n \in \mathbb{N} - \{0\}$ and $g(\infty) = \infty$. Then the associated cost function is given by $g^*(0) = 0, g^*(1) = 1$ and $g(n) = \infty$ for all $n \in \overline{\mathbb{N}}$ with $n \geq 2$. As discussed this is the cost function which gives classical domination.*

The dual is also the case but this requires proving.

Theorem 3.37. *If $k : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ is a cost function, then k has a right adjoint and this right adjoint is a scaling function.*

Proof. The element ∞ makes $\overline{\mathbb{N}}$ closed under nonempty joins and the empty join is $0 \in \overline{\mathbb{N}}$.

We must then check that k preserves joins. We have $k(0) = 0$, so the empty join is preserved. Next consider $k(\sup A)$. If A is a bounded set then $\sup(A) = \max(A)$ and so $k(\sup(A)) = k(\max(A)) = \max(k(A)) = \sup(k(A))$. If A is unbounded then $\sup(A) = \infty$ and so $k(\sup(A)) = k(\infty) = \sup(k(\mathbb{N})) = \sup(k(A))$ where the last equality holds since for each element in $k(A)$ there is one in $k(\mathbb{N})$ larger than it since $\mathbb{N}$ is unbounded, and for each element in $k(\mathbb{N})$ there is one in $k(A)$ larger than it because A is unbounded.

We now show that k_* is a scaling function. Certainly it is the case that k_* is nondecreasing and $k_*(0) = 0$. Thus it remains to check that $k_*(\infty) = \infty$. We have that $k_*(\infty)$ is the largest value b making $k(b) \leq \infty$. But $k(\infty) \leq \infty$ and ∞ is the largest element of $\overline{\mathbb{N}}$, thus $k_*(\infty) = \infty$. □

Example 3.38. *If k is a cost function, and more specifically, k is the identity, then k_* is the identity also. As already mentioned classical broadcast domination can be viewed as S-cast domination where the graph G is equipped with the identity scaling function.*

These theorems taken together give us a bijection between cost and scaling functions. Consequently from this point on we will consider (G, g) as the object of study, which we think of as a graph equipped with a scaling function g and its adjoint cost function g^*.

Given a scaling function g or a cost function k we can ask what their respective closed elements are. Since every cost function is the left adjoint of a scaling function we won't consider this as two separate questions.

Proposition 3.39. *Let g be a scaling function and g^* its associated cost function. The closed elements with respect to the closure operator $g \circ g^*$ are precisely the elements $x \in \overline{\mathbb{N}}$ with $g^*(x+1) > g^*(x)$. The closed elements with respect to the interior operator $g^* \circ g$ are precisely the elements $x \in \overline{\mathbb{N}}$ such that $g(x) < g(x+1)$.*

Proof. Assume that x is closed with respect to $g \circ g^*$. We know $g(g^*(x+1)) \geq x+1$. Since $g^*(g(x)) = x$ it follows that $g^*(x+1) > g^*(x)$. Conversely suppose $g^*(x+1) > g^*(x)$. We know $g^*(x)$ is the largest element making $g(g^*(x)) \geq x$ which can then easily be seen to be x itself.

The proof of the result concerning the interior operator is similar. □

One of the reasons broadcast domination seems more well behaved in many respects than classical domination (like the broadcast domination problem lying in $\mathcal{P}$) is because the scaling function underlying broadcast domination is superadditive, but in the classical case it is not.

Definition 3.40. *Let $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ be a non-decreasing function. We say*

- *g is superadditive if $g(x+y) \geq g(x) + g(y)$.*
- *g is subadditive if $g(x+y) \leq g(x) + g(y)$.*

Addition on $\overline{\mathbb{N}}$ agrees with addition on $\mathbb{N}$ for all finite numbers and $a + \infty = \infty = \infty + a$.

Example 3.41. *Linear functions ($g(x) = kx$) are both superadditive and subadditive.*

Example 3.42. *The scaling function given by $g(x) = 2^x$ is superadditive.*

Example 3.43. *The cost function given by $k(x) = \lceil \sqrt{x} \rceil$ is subadditive.*

Proposition 3.44. *A scaling function g is superadditive if and only if the cost function g^* is subadditive.*

Proof. Let g be superadditive. Then $g(g^*(x) + g^*(y)) \geq g(g^*(x)) + g(g^*(y)) \geq x + y$. We know that $g^*(x+y)$ is the least element a with $x+y \leq g(a)$, thus $g^*(x+y) \leq g^*(x)+g^*(y)$.

Let g^* be subadditive. Then $g^*(g(x)+g(y)) \leq g^*(g(x))+g^*(g(y)) \leq x+y$. We know that $g(x+y)$ is the largest value making $g^*(g(x+y)) \leq x+y$. But $g^*(g(x)+g(y)) \leq x+y$, thus $g(x+y) \geq g(x)+g(y)$. □

It is not in general the case that if g is subadditive then that g^* is superadditive.

Example 3.45. *Let $g : \{0, 1, 2\} \to \{0, 1, 2, 3\}$ be nondecreasing with*

$$g(0) = 0,$$
$$g(1) = 2,$$
$$g(2) = 2.$$

It is easy to verify that g is subadditive. The left adjoint of g, $g^* : \{0, 1, 2, 3\} \to \{0, 1, 2, 3, 4\}$ is given by

$$g^*(0) = 0,$$
$$g^*(1) = 1,$$
$$g^*(2) = 1.$$

since $g^*(2) < g^*(1) + g^*(1)$, g^* is not superadditive.

3.5 Cost and Scaling related

In the preceding section we saw that the cost and scaling frameworks are related by an adjunction. In this section we strengthen this relationship first by showing that an adjunction exists between the poset of S-casts and the poset of broadcasts (assuming a scaling function has been fixed). This adjunction is inherited from the one already discussed. We then show that the broadcast domination numbers in each framework agree for all graphs.

Lemma 3.46. *Let (G, g) be a graph equipped with a scaling function g. If h is an S-cast on (G, g), then $g^* \circ g \circ h \leq h$. If f is a broadcast on G then $f \leq g \circ g^* \circ f$.*

Proof. By Proposition 3.39, $g \circ g^*$ is a closure operator. Thus it follows immediately that $(g \circ g^*)(f(v)) \geq f(v)$ for all $v \in V(G)$ and consequently that $f \leq g \circ g^* \circ f$.

Using the fact that $g^* \circ g$ is an interior operator and a similar argument we get that $g^* \circ g \circ h \leq h$. □

Proposition 3.47. *Let (G, g) be a graph equipped with a scaling and cost function. Then there exists an adjunction between the set A of all S-casts on (G, g) and the set B of all broadcasts on (G, g) (both sets equipped with the pointwise order). This adjunction consists of $F : B \to A$, $F(f) = g^* \circ f$ and $G : A \to B$, $G(h) = g \circ h$.*

Proof. First assume $F(f) \leq h$. Thus $g^* \circ f \leq h$. Since g is nondecreasing we have that $g \circ g^* \circ f \leq g \circ h$. By Lemma 3.46 we have that $f \leq g \circ g^* \circ f \leq g \circ h = G(h)$.

Next assume $f \leq G(h)$. Thus $f \leq g \circ h$. Since g^* is nondecreasing $g^* \circ f \leq g^* \circ g \circ h$ and by Lemma 3.46 $F(f) = g \circ f \leq g \circ g^* \circ h \leq h$. □

Since we have a Galois connection between the set of S-casts and the set of broadcasts, we now ask which S-casts and broadcasts are closed.

It is easy to verify that a minimal dominating S-cast is closed with respect to $g^* \circ g$. Let h be a minimal dominating broadcast on G. Consider $g \circ h$, the broadcast induced by h. Since h is minimal dominating it follows that $g(h(v) - 1) \leq g(h(v))$. Thus applying Proposition 3.39 gives us that $(g^* \circ g \circ h)(v) = h(v)$ for all $v \in V(G)$ and consequently that h is closed. We write $\overline{h}$ to refer to the closure of a function.

In this proof, we did not use the fact that h was dominating and so this calls for a slightly more general notion.

Definition 3.48. *Let (G, g) be a graph equipped with a scaling function and a cost function. Then we say an S-cast h is minimal if $g(h(v)-1) < g(h(v))$, we say an S-cast is maximal if $g(h(v)+1) > g(h(v))$. We say f is a minimal broadcast if $g^*(f(v)-1) < g^*(f(v))$ and that f is a maximal broadcast if $g^*(f(v)+1)) > g^*(f(v))$.*

Proposition 3.49. *Let (G, g) be a graph equipped with a cost and scaling function. An S-cast h is closed with respect to $g^* \circ g$ if and only if h is a minimal S-cast. A broadcast f is closed with respect to $g \circ g^*$ if and only if it is a maximal broadcast.*

Proof. Assume h is an S-cast closed with respect to $g^* \circ g$. Then for each $v \in V(G)$, $g^*(g(h(v))) = h(v)$. Applying Proposition 3.39 we get that $g(h(v)-1) < g(h(v))$ which implies that h is minimal. If h is minimal then $g(h(v)-1) < g(h(v))$ for each $v \in V(G)$ which means $h(v)$ is a closed element with respect to $g \circ g^*$ for each $v \in V(G)$. This in turn means that h is closed.

Next assume f is a broadcast closed with respect to $g \circ g^*$. Then for each $v \in V(G)$, $g(g^*(f(v))) = f(v)$, which again applying Proposition 3.39 implies that for each $v \in V(G)$, $g^*(f(v)+1) > g^*(f(v))$. If f is maximal then $g^*(f(v)+1) > g^*(f(v))$ for each $v \in V(G)$ which means that $f(v)$ is closed for each $v \in V(G)$ with respect to $g^* \circ g$ which as before means that f is closed. □

Proposition 3.50. *Let (G, g) be a graph equipped with cost and scaling functions. Then if h is an S-cast, then $(g \circ h)(v) = (g \circ \overline{h})(v)$. If f is an broadcast then $(g^* \circ f)(v) = (g^* \circ \overline{f})(v)$.*

Proof. Let h be an S-cast. We have that $g^*((g \circ h)(v))$ is the smallest element b such $g(b) \geq (g \circ h)(v)$, but we know there exists a b such that $g(b) = (g \circ h)(v)$ in particular $h(v)$, hence $(g \circ \overline{h})(v) = g(g^* \circ g \circ h)(v)) = (g \circ h)(v)$.

The result for broadcasts follows similarly. □

Theorem 3.51. *For any scaling function g and any graph G*

$$\gamma_s^g(G) = \gamma_c^{g^*}(G).$$

Proof. Let f be an optimal dominating broadcast. Consider $g^* \circ f = h$. Trivially, $g^* \circ f \leq h$. Thus applying Proposition 3.47 we get that $f \leq g \circ h$. This gives us that h

is a dominating S-cast. Furthermore $w_s^g(h) = \sum_{v \in V} h(v) = \sum_{v \in V}(g^* \circ f)(v) = \gamma_c^{g^*}(G)$. Thus $\gamma_s^g(G) \leq \gamma_c^{g^*}(G)$.

Furthermore if h is an optimal dominating S-cast, consider $g \circ h = f$. Since h is dominating, by definition f is dominating. Furthermore h is minimal and by Proposition 3.49 closed. Thus $\sum_{v \in V}(g^* \circ g \circ h)(v) = \sum_{v \in V} h(v)$. It follows that $\gamma_c^{g^*}(G) \leq \gamma_s^g(G)$. □

Corollary 3.52. *Let (G, g) be a graph equipped with a scaling function g. If h is an optimal dominating S-cast, then $g \circ h$ is an optimal dominating broadcast.*

Proof. Let h be an optimal dominating S-cast. Consequently h is minimal. Thus we invoke Lemma 3.46 to get $g^* \circ g \circ h = h$. Then by Theorem 3.51 we have $\gamma_c^{g^*}(G) = \gamma_s^g(G) = \sum h(v) = \sum g^*(g \circ h(v))$ which means $g \circ h$ is an optimal dominating broadcast. □

Just as $g \circ h$ can be thought of as the broadcast associated to h, $g^* \circ f$ can be thought of as the associated S-cast to a broadcast f. This does not mean that $g \circ g^* \circ f = f$.

Just because the domination numbers agree in each framework doesn't mean all graph parameters will agree. For instance a difference can be noted while studying efficiency on (G, g). It is known that for classical domination not every graph has an efficient dominating set [3]. Furthermore we can emulate classical domination on any graph with the scaling function $g(0) = 0, g(x) = 1, g(\infty) = \infty$ for all $x \in \mathbb{N}$. Thus we can find a (G, g) where no efficient dominating S-casts exist. It is shown in Section 4 that for broadcasts on (G, g), efficient dominating broadcasts always exist.

4 Efficiency

We look at efficiency in the case of S-casts and broadcasts separately. Despite the two theories being heavily related it's not immediate that efficiency will agree in both systems.

4.1 Efficient S-casts

In this section we prove some structural results relating to efficient S-casts and broadcasts.

Definition 4.1. *Let (G, g) be a graph equipped with a scaling function g. A broadcast f on (G, g) is efficient if $N_f[v] \cap N_f[u] = \emptyset$ for all $v, u \in V_f^+$ where $v \neq u$.*

Definition 4.2. *Let (G, g) be a graph equipped with a scaling function g. We say that an S-cast h on (G, g) is efficient if $g \circ h$ is an efficient broadcast.*

Lemma 4.3. *Let (G, g) be a graph equipped with a superadditive scaling function. Then for any dominating S-cast h, there exists an h' with $w_s^g(h') = w_s^g(h)$ and h' is dominating and independent and $V_{h'}^+ \subseteq V_h^+$.*

Proof. Let h be a dominating S-cast on (G, g). Assume it is not independent. Then there exist two vertices $v, u \in V_h^+$ with $v \in N_h(u)$ and $v \neq u$.

Now construct h' with $h'(w) = w$ if $w \neq u, v$ and $h'(u) = h(v) + h(u)$ and $h'(v) = 0$. Because g is superadditive $(g \circ h')(u) \geq (g \circ h)(v) + (g \circ h)(u)$ and so any vertex $w \in N_h(v)$ is also in $N_{h'}(u)$ and thus h' is dominating. Furthermore $V_{h'}^+ \subsetneq V_h^+$ and $V_{h'}^+$ is finite, so if h' is not independent we can repeat this procedure a finite number of times and be guaranteed an independent S-cast. $\square$

Lemma 4.4. *Let (G, g) be a graph equipped with a superadditive scaling function g. Let h be an inefficient dominating S-cast. There exists an efficient dominating S-cast h' such that $|V_{h'}^+| < |V_h^+|$ and where $w_s^g(h') = w_s^g(h)$.*

Proof. Let t be an S-cast which is dominating but not efficient. Then by Lemma 4.3 there is an independent S-cast h such that $V_h^+ \subseteq V_t^+$ and with $w_s^g(h) = w_s^g(t)$. If h is efficient we are done. Assume h is not efficient. Then there are vertices $v, w \in V_h^+$ with

$v \neq w$ and $u \in V(G)$ such that $u \in N_h[v] \cap N_h[w]$. Since h is independent, $u \neq v, w$. It follows that there exists a path P from v to w passing through u with length less or equal to $(g \circ h)(v) + (g \circ h)(w)$. Let x be a vertex in P distance $(g \circ h)(w)$ from v.

Consider now a new S-cast h' such that $h'(z) = h(z)$ for $z \notin \{x, v, w\}$, $h'(v) = 0 = h(w)$ and $h'(x) = h(v) + h(w)$. Note that $w_s^g(h') = w_s^g(h) = w_s^g(t)$ and $|V_{h'}^+| < |V_h^+| \leq |V_t^+|$. The fact that h' dominates follows from the superadditivity of g. Since $g(h(v) + h(w)) \geq g(h(v)) + g(h(w))$ it follows immediately that $N_h(v) \subseteq N_{h'}(x)$ and $N_h(w) \subseteq N_{h'}(x)$.

It may not be that h' is efficient, but V_h^+ is finite and $|V_{h'}^+| < |V_h^+|$, and so if we repeat the procedure mentioned in this proof the process will eventually stop and leave us with an efficient optimal dominating broadcast.

□

Theorem 4.5. *Let g be a scaling function. Then (G, g) has an efficient $\gamma_s^g(G)$ dominating S-cast for all G if and only if g is superadditive.*

Proof. Assume g is superadditive.

Let h be an optimal dominating S-cast. If h is efficient we are done, if h is not efficient we apply Lemma 4.4 to find an optimal dominating S-cast which is.

Assume g is not superadditive.

We construct a graph G such that (G, g) does not have an efficient optimal dominating broadcast. Since g is not superadditive there exist natural numbers m and n such that $g(m+n) < g(m)+g(n)$. Pick x to be the smallest m for which the above inequality holds for some n. Then pick y to be the smallest number such that $g(x + y) < g(x) + g(y)$. Note that this means that $g(x - 1) < g(x)$ as if $g(x - 1) = g(x)$ then we would have $g((x - 1) + y) \leq g(x + y) < g(x) + g(y) = g(x - 1) + g(y)$ contradicting that fact that x was chosen to be minimal. A similar argument gives that $g(y - 1) < g(y)$.

We construct the graph in the following way. Glue $x + y + 1$ copies of $P_{g(x)+1}$ together at an end vertex, i.e. a vertex of degree one. Call this graph G_x. Construct G_y in the same way except with $x + y + 1$ copies of $P_{g(y)+1}$. Now construct G by gluing G_x and G_y together at an end vertex.

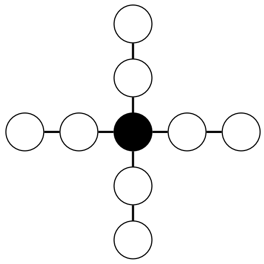

Figure 9: Let g be the scaling function mentioned in Example 4.6. Then this graph represents G_x and G_y as described in Theorem 4.5 obtained by gluing 4 copies of P_3 together at an end vertex.

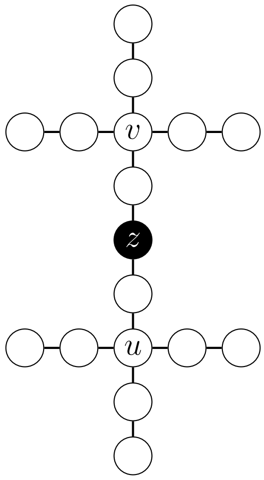

Figure 10: Let g be the scaling function mentioned in Example 4.6. Then this graph represents G as described in Theorem 4.5 obtained by gluing G_x to G_y by an end vertex (as seen in Figure 9).

Let v be the centre of G_x and u the centre of G_y and z the vertex that attaches G_x and G_y. Then let h be the S-cast given by

$$h(s) = \begin{cases} 0 & \text{if } s \notin \{u, v\} \\ x & \text{if } s = v \\ y & \text{if } s = y. \end{cases}$$

It is clear that this h dominating. Furthermore $z \in N_h(v)$ and $z \in N_h(u)$, so h is not efficient. We now show that $\gamma_s^g(G) = x+y$ and then that h is the only optimal dominating broadcast.

Let h' be a minimal dominating S-cast with $w_s^g(h') \leq x + y$. Let $u_1, \ldots, u_{x+y}$ refer to the end vertices closest to u and $v_1, \ldots, v_{x+y}$ the end vertices closest to v. Note that the diameter of G is $d_G(u_i, v_j) = 2g(x) + 2g(y)$ where i, j are some natural numbers satisfying $1 \leq i, j \leq x + y$. Consider a $u_i - v_j$ path P and a vertex w distance $g(x) + g(y)$ from v_j. Then $d_G(w, u_i) = g(x) + g(y)$. We can conclude that w is a central vertex and that $\mathrm{rad}(G) = g(x) + g(y)$ as if the radius were less than $g(x) + g(y)$ then there would be a $u_i - v_j$ path of length less than $2g(x) + 2g(y)$. Furthermore the distance from w to any other leaf vertex is $g(x) + g(y)$ by symmetry.

Since $g(x+y) < g(x) + g(y)$ it is not possible for h' to be a radial S-cast. Thus $|V_{h'}^+| > 1$. We now show that $|V_{h'}^+| = 2$. To do so we show that any vertex $x \in V_{h'}^+$ dominates either all of $v_1, \ldots, v_{x+y}$ or all of $u_1, \ldots, u_{x+y}$.

It is not possible for each of the end vertices $v_1, \ldots, v_{x+y}$ to be dominated by a distinct vertex as there are $x + y$ of these end vertices and $w_s^g(h') \leq x + y$ and this would leave no vertices to dominate the other set of end vertices without the cost of h' exceeding $x + y$. Now suppose that $s \in V_{h'}^+$, that $v_i, v_j \in N_{h'}(s)$ that $t \in V_{h'}^+$ and that $u_{i'}, u_{j'} \in N_{h'}(t)$. (Such a t can be shown to exist by a symmetric argument.) Let $d_G(s, v) = k$. Then $(g \circ h')(s) \geq g(x) + k$ as either the $s - v_i$ path or the $s - v_j$ path passes through v. This implies that $\{v, v_1, \ldots, v_{x+y}\} \subseteq N_{h'}(s)$. A similar argument gives that $\{u, u_1, \ldots, u_{x+y}\} \subseteq N_{h'}(t)$. It follows that $V(G) \subseteq N_{h'}(s) \cup N_{h'}(t)$ and so $V_{h'}^+ = \{s, t\}$.

Assume $d_G(s, v) = k \neq 0$. Then $h'(s) > x$ as $(g \circ h')(s) > g(x)$. Thus $h'(t) < y$ as $w_s^g(h') = x + y$. Then $(g \circ h')(t) < g(y)$ (since $g(y-1) < g(y)$) which contradicts the fact that t h-dominates the leaves $u_1, \ldots, u_{x+y}$. Thus $s = v$ and a symmetric argument gives $t = u$. If $h(v) < x$ then the leaves $v_1, \ldots, v_{x+y}$ are not h-dominated by v and similarly if $h(t) < y$ then the leaves $u_1, \ldots, u_{x+y}$ are not h-dominated by u. Hence $h' = h$ and so the only optimal dominating S-cast on (G, g) is inefficient. □

Example 4.6. *Let g be a scaling function given by*

$$g(x) = \begin{cases} 0 & x \leq 3 \\ 3 & 4 \leq x \leq 7 \\ 5 & x = 8 \\ \infty & x \geq 9. \end{cases}$$

Then g is not superadditive as $g(4+4) < g(4) + g(4)$. If we were to pick x and y as in the proof $x = 4 = y$.

4.2 Broadcast Efficiency

The above S-cast results have analogues in the case of broadcasts with a cost function.

Lemma 4.7. *Let (G, g) be a graph equipped with a subadditive cost function. Let f be an inefficient dominating broadcast. Then there exists an efficient broadcast f' such that $|V_{f'}^+| < |V_f^+|$ and where $w_c^{g*}(f') \leq w_c^{g^*}(f)$.*

Proof. Let f be an inefficient broadcast. Since $f \leq g \circ g^* \circ f$ it follows that $g^* \circ f$ is an inefficient S-cast. Since g^* is subadditive, g is superadditive so we can apply Lemma 4.4 to find an efficient dominating S-cast h with $|V_h^+| \leq |V_{g^* \circ f}^+|$ and where $w_s^g(h) = w_s^g(g^* \circ f)$. Let $f' = g \circ h$. It is clearly efficient and dominating. We have that $|V_{f'}^+| = |V_h^+| < |V_{g^* \circ f}^+| = |V_f^+|$. Furthermore $w_c^{g^*}(f') = w_s^g(g^* \circ g \circ h) < w_s^g(h) = w_c^{g^*}(f)$. $\square$

Corollary 4.8. *If g^* is a subadditive cost function, then there exists an optimal dominating broadcast f which is efficient.*

5 Some Early S-cast and Cost Domination Results

Most bounds on the S-cast domination number are determined by the associated cost function (its left adjoint). Here we generalise the early broadcast results, giving bounds for the S-cast domination number and the upper S-cast domination number.

Proposition 5.1. *For a graph G and a scaling function g the following always holds:*

$$\gamma_s^g(G) \leq \min\{\gamma(G)g^*(1),\ g^*(rad(G))\}.$$

Proof. First we show there exists a dominating S-cast h with $\sum_{v \in V} h(v) = \gamma(G)g^*(1)$. Let $S \subseteq V(G)$ be a dominating set with $\gamma(G)$ vertices. It follows that for each $v \in V(G)$, there exists a $u \in S$ such that $d_G(v, u) \leq 1$. Consider the S-cast h with $h(v) = g^*(1)$ for $v \in S$ and $h(v) = 0$ otherwise. Thus $g \circ h(v) \geq 1$ for each $v \in S$ and so it follows that $g \circ h$ is a dominating broadcast.

If we let $v \in V(G)$ be a vertex in the centre of G then let h be an S-cast with $h(v) = g^*(\mathrm{rad}(G))$ and $h(u) = 0$ for all other vertices $u \neq v$. We see that $g \circ h$ is a dominating broadcast. □

It is possible that $g(a) < \mathrm{rad}(G)$ for all $a \in \mathbb{N}$. In this case $g^*(\mathrm{rad}(G)) = \infty$. So the result still holds, but no longer provides any useful information.

Proposition 5.2. *Let (G, g) be a graph equipped with a scaling function. Then*

$$g^*(diam(G)) \leq \Gamma_s^g(G).$$

Proof. Let $v \in V(G)$ be a vertex on the periphery of G. Consider an S-cast h where $h(v) = g^*(\mathrm{diam}(G))$ and $h(u) = 0$ for all vertices $u \neq v$. It is minimal dominating and so combining what we know we get $w_s^g(h) = g^*(\mathrm{diam}(G)) \leq \Gamma_s^g(G)$. □

Proposition 5.3. *For a graph G and a scaling function g, where $g^*(1) = 1$, we have*

$$\Gamma(G) \leq \Gamma_s^g(G).$$

Proof. We must show that there exists a minimal dominating S-cast h with $\sum_{v \in V} h(v) = \Gamma(G)$.

First let S be a minimal dominating set with $\Gamma(G)$ vertices. Then define $h(v) = g^*(1) = 1$ if $v \in S$ and $h(v) = 0$ otherwise. It is clear that h is a dominating S-cast. Furthermore if $h(v)$ were reduced for any v then $g \circ h$ would no longer be dominating. $\square$

Below we generalise the bound $\Gamma_b(G) \leq m$ given in [8] to the S-cast case. Unfortunately it becomes quite a bit more technical and requires some computation.

We introduce something called the set of y-partitions of x. For $x, y \in \mathbb{N}$, we define $p_y(x)$ to be the set of all nonincreasing functions $f : \underline{x-1} \to \underline{y}$ with $\sum_{a \in \underline{x-1}} f(a) = x$. It captures the unique ways to partition x as a sum where no term is greater than y.

Example 5.4. *The set $p_2(4)$ contains three functions:*

f_1 with $f_1(0) = 1, f_2(1) = 1, f_1(2) = 1, f_1(3) = 1$,

f_2 with $f_2(0) = 2, f_2(1) = 1, f_2(2) = 1, f_2(3) = 0$ and

f_3 with $f_3(0) = 2, f_3(1) = 2, f_3(2) = 0, f_3(3) = 0$.

Theorem 5.5. *Let G be a connected graph with $\operatorname{diam}(G) = d$ and m edges and let g a scaling function. Then*

$$\Gamma_s^g(G) \leq \max\{\sum_{x \in \underline{m-1}} g^*(k(x)) : k \in p_d(m)\}$$

where $p_d(m)$ is the set of $\operatorname{diam}(G)$-partitions of m.

Proof. Let $d = \operatorname{diam}(G)$. Consider a minimal dominating S-cast h. Because $g \circ h$ need not be a minimal dominating broadcast, we consider f, a minimal dominating broadcast smaller than $g \circ h$. Note that if $g \circ h(v) = 0$ then $f(v) = 0$ and if $g \circ h(v) \neq 0$ then $f(v) \neq 0$, as otherwise h would not have been a minimal S-cast. Consequently $V_f^+ = V_{g \circ h}^+ = V_h^+$ by Lemma 3.24.

Since h is minimal, $h(v) = g^*((g \circ h)(v)) = g^*(f(v))$ for each $v \in V_h^+$ by Proposition 3.49.

We know that $\sum_{v \in V_f^+} f(v) \leq m$ by Theorem 2.11 and since each $f(v) \leq \operatorname{diam}(G)$, there must exist a $f' \in p_d(m)$ with $f'(i) \geq f(v_i)$ where we number the vertices of V_f^+ starting from 0 such that $f(v_i) \geq f(v_{i+1})$. Then $\sum_{v \in V_f^+} h(v) = \sum_{v \in V_f^+} g^*(f(v)) \leq \sum_{i \in \underline{m-1}} g^*(f'(i))$. Now since $f' \in p_d(m)$ it follows that $\sum_{v \in V_f^+} h(v) \leq \max\{\sum_{x \in \underline{m}} g^*(k(x)) \mid$

$k \in p_d(m)\}$. Since this holds for each minimal S-cast h we get that

$$\Gamma_s^g(G) \leq \max\{\sum_{x \in \underline{m-1}} g^*(k(x)) \mid k \in p_d(m)\}.$$

$\square$

Corollary 5.6. *Let G be a graph with m edges and let g be a scaling function such that g^* is superadditive. Then $\Gamma_s^g(G) \leq g^*(m)$.*

Proof. We know by Theorem 5.5 that $\Gamma_s^g(G) \leq \max\{\sum_{x \in \underline{m-1}} g^*(k(x)) \mid k \in p_d(m)\}$. Let $\mathrm{diam}(G) = d$ and $k \in p_d(m)$. Then since g^* is superadditive we have that $g^*(m) = g^*(\sum_{x \in \underline{m-1}} k(x)) \geq \sum_{x \in \underline{m-1}} g^*(k(x))$. Hence $g^*(m) \geq \max\{\sum_{x \in \underline{m-1}} g^*(k(x)) : k \in p_d(m)\}$ and the result follows. $\square$

We have not yet related either of the S-cast domination numbers to the broadcast domination numbers.

Proposition 5.7. *Let G be a graph. For any $t \in \mathbb{N}$ there exists a scaling function g such that $\gamma_s^g(G) = \Gamma_s^g(G) = t$.*

Proof. Fix t and consider a scaling function g with $g(x) = 0$ for $x < t$ and $g(t) = \mathrm{diam}(G)$. Note that g is a scaling function and that for any minimal S-cast h, we have $\sum v \in V h(v) = t$. $\square$

However applying our adjoints we get

Theorem 5.8. *For a graph G equipped with scaling function g we have*

$$\gamma_s^g(G) \leq \max\{\sum_{x \in \underline{\gamma_b(G)-1}} g^*(k(x)) \mid k \in p_d(\gamma_b(G))\}.$$

Proof. Let f be a $\gamma_b(G)$ broadcast. Then consider the S-cast $h(v) = g^*(f(v))$. Certainly $g \circ h(v) \geq f(v)$ for each v so h is a dominating S-cast. Furthermore it is clear that $\sum_{v \in V} h(v) \leq \max\{\sum_{x \in \underline{\gamma_b(G)-1}} g^*(k(x)) \mid k \in p_d(\gamma_b(G))\}$, since each $f(v) \leq \mathrm{diam}(G)$ and $\sum_{v \in V} f(v) = \gamma_b(G)$. $\square$

Corollary 5.9. *Let G be a graph and let g be a scaling function such that g^* is superadditive. Then $\gamma_s^g(G) \leq g^*(\gamma_b(G))$.*

Proof. Let $\operatorname{diam}(G) = d$. We know by Theorem 5.5 that $\gamma_s^g(G) \leq \max\{\sum_{x \in \underline{\gamma_b(G)-1}} g^*(k(x)) \mid k \in p_d(\gamma_b(G))\}$. Let $k \in p_d(\gamma_b(G))$. Then since g^* is superadditive we have that $g^*(\gamma_b(G)) = g^*(\sum_{x \in \{1,2,\dots,\gamma_b(G)\}} k(x)) \geq \sum_{x \in \underline{\gamma_b(G)-1}} g^*(k(x))$. Hence

$$g^*(\gamma_b(G)) \geq \max\{\sum_{x \in \underline{\gamma_b(G)-1}} g^*(k(x)) \mid k \in p_d(\gamma_b(G))\}$$

and the result follows. □

Definition 5.10. *A scaling function g is strictly superadditive if $g(x+y) > g(x) + g(y)$.*

Proposition 5.11. *Let G be a graph and g a strictly superadditive scaling function. Then $\gamma_s^g(G) = g^*(\operatorname{rad}(G))$.*

Proof. Let h be an efficient optimal dominating S-cast on (G, g). Let $u, v \in V_h^+$ be two vertices such that $N_G(N_h[v]) \cap N_h[u] \neq \emptyset$. Then $d_G(v, u) = (g \circ h)(v) + (g \circ h)(u) + 1$. Let P be a $v - u$ geodesic and let $x \in P$ be a vertex such that $d_G(v, x) = (g \circ h)(u)$. Thus $d_G(u, x) = (g \circ h)(v) + 1$. Let h' be an S-cast such that $h'(x) = h(v) + h(u)$, $h'(v) = 0 = h'(u)$ and $h'(w) = h(w)$ for all other $w \in V(G)$. To show that h' is dominating we just need to check that the vertices in $N_h[v] \cup N_h[u]$ are dominated by h'. If $w \in N_h[v]$ then $d_G(x, w) \leq (g \circ h)(u) + (g \circ h)(v) \leq g(h(v) + h(u))$. Thus w is dominated by x. If $w \in N_h[u]$ then $d_G(x, w) \leq (g \circ h)(u) + (g \circ h)(v) + 1 \leq g(h(v) + h(u))$. Thus w is dominated by x. Thus h' is dominating and has the same cost as h.

This process can be repeated until there is just a single vertex left. Thus a radial broadcast is optimal dominating and so $\gamma_s^g(G) = g^*(\operatorname{rad}(G))$. □

Proposition 5.12. *Let $g : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ be a scaling function of the form $g(n) = \lfloor \frac{n}{k} \rfloor$ for $k \in \mathbb{N}$. Then for any graph G we have*

$$\gamma_s^g(G) = k\gamma_b(G).$$

Proof. Given an optimal dominating broadcast f, we construct an S-cast h with $w_s^g(h) = k\gamma_b(G)$. Let $h(v) = g^*(f(v)) = kf(v)$. It is clear that $g \circ h = f$. Furthermore it is clear $\sum_{v \in V} h(v) = k\gamma_b(G)$.

Now it remains to verify that there is no dominating S-cast h with $\sum_{v \in V} h(v) < k\gamma_b(G)$. Assume there exists an S-cast h with $\sum_{v \in V} h(v) = t < k\gamma_b(G)$. Then $\sum_{v \in V} g \circ h(v) =$

$\sum_{v \in V} \lfloor \frac{h(v)}{k} \rfloor \leq \lfloor \frac{t}{k} \rfloor < \gamma_b(G)$. But then $g \circ h$ is a broadcast with cost less than $\gamma_b(G)$, which is a contradiction. $\square$

This is a very natural example of a scaling function where there is a fixed cost for a discrete improvement.

6 Complexity Theory

Before we can tackle computational problems related to S-casts we must introduce some concepts from complexity theory. In this section we introduce complexity classes, Turing machines and oracles. For a full treatment see [19].

6.1 Complexity Classes

In order to create a formal framework in which to study computational problems we fix an alphabet Σ and consider strings of letters from the alphabet, which we call Σ-*strings*. Intuitively we think of certain strings of characters as encoding certain mathematical statements. We let Σ^* denote the set of all finite strings of characters from Σ.

Definition 6.1. *A language L is any nonempty set $L \subseteq \Sigma^*$.*

The languages we use generally define a computational problem and in fact we may use the term *problem* interchangeably with *language*. For instance, a language L we might consider is the set of all strings which encode (G, k) where G is an arbitrary graph, $k \in \mathbb{N}$ and G has a dominating set of size k. Note that G and k can be varied in the above example and so we can ask for which G and k we have $(G, k) \in L$.

To make the above example more general we can think of a language L as defining a *decision problem*, where it must be decided whether or not a string x belongs to the language L. A computational problem can often be converted easily into a decision problem. For instance the problem of finding the largest prime factor of a number n, can be converted into the decision problem of when given a pair (p, k, n) determining whether n has a prime factor $p \geq k$.

Definition 6.2. *A Complexity Class $\mathcal{C}$ is a set of languages.*

A number of complexity classes are intuitively defined in terms of how quickly a string can be determined to be a member of a language L. In order to formalise this concept we introduce the *Turing machine* as a model for computation.

6.2 Turing machines

Introduced by Alan Turing in [21], *Turing machines* are a way to formalise the concept of computability. There are number of variations of Turing machines, we follow the treatment in [19].

Intuitively a Turing machine can be thought of as a tape infinitely long in one direction partitioned into discreet units (or squares) which can either be *blank* or contain a symbol from some alphabet Σ. It has a number of *states* it can be in. It also has a *head* which is at a certain position on the tape and which reads the symbol at that position. The head then decides whether to write a new symbol in its current position, whether or not to change state and whether to move left or right. This process continues until a state is entered which indicates that the computation is complete. The machine may potentially never enter this state.

To make this idea formal we consider the following components of a Turing machine. We have an alphabet Σ and set $T = \Sigma \cup \{b\}$, $b \notin \Sigma$ of symbols which can appear on the tape (the symbol b is the blank symbol), Q the set of states (with distinguished states $q_0, q_{accept}, q_{reject} \in Q$) and finally a transition function $\delta : Q \times T \to Q \times T \times \{L, R\}$. If $\delta(q_x, t_x) = (q_y, t_y, R)$, which means that the machine is in state q_x and the head is currently over a symbol t_x, then the machine changes it state from q_x to q_y, writes the symbol t_y in its current location and then the head moves one square to the right (if instead $\delta(q_x, t_x) = (q_y, t_y, L)$ then the machine would do the same as above except that it would move one square to the left). Each application of the transition function we call a step. The machine starts in state q_0 and the computation terminates only when it enters either the state q_{accept} or q_{reject}.

To summarise the above we give the following definition.

Definition 6.3. *A Turing machine is a 7-tuple* $(Q, \Sigma, T, \delta, q_0, q_{accept}, q_{reject})$ *where*

1. *Q is the finite set of states,*
2. *Σ is the finite set of symbols that can be written onto the tape,*
3. *$T = \Sigma \cup \{b\}$, $b \notin \Sigma$, is the finite set of symbols which can appear on the tape,*

4. $\delta : Q \times T \to Q \times T \times \{L, R\}$ *is the transition function,*

5. q_0 *is the state the machine starts in,*

6. q_{reject} *is the reject state,*

7. q_{accept} *is the accept state and* $q_{accept} \neq q_{reject}$.

Consider a Turing machine M and a tape infinite in one direction. The Turing machine receives a finite input string $w = w_0 w_1 \ldots w_n$ where $w_i \in \Sigma$ for $0 \leq i \leq n$. The input is placed on the leftmost $n + 1$ squares of the tape with w_0 being placed on the first square, w_1 on the second square, etc. All the other squares begin with the blank symbol b. The machine begins in state q_0 and the head begins at the first square. The transition function thus begins with $\delta(q_0, w_0)$ and then depending on the output writes a symbol, changes state and moves either to the left or right. If the head would move off the leftmost end of the tape, it stays where it is instead. This process terminates only when the state q_{accept} or q_{reject} is entered. Given an input w we say M accepts w if M ends in the state q_{accept} and we say M rejects w when it ends in the state q_{reject}. When a Turing machine M enters a q_{accept} or q_{reject} state we say that M has halted.

Given a language L we may ask if there is a Turing machine M_L which accepts only strings $x \in L$ and rejects all strings $x \notin L$.

Definition 6.4. *A language* L *is decidable if there exists a Turing machine* M_L *which accepts all strings* $x \in L$ *and rejects all strings* $x \notin L$*. We say that* M_L *decides* L*.*

We only consider decidable languages from here on.

Many of the most important complexity classes are defined in terms of how quickly it can be determined whether or not a string belongs to a particular language. This time is measured in terms of how many steps a Turing machine takes to halt. Given a Turing machine M, the *running time* of M is given by a function $f : \mathbb{N} \to \mathbb{N}$, where f returns the maximum number of steps M takes to halt, on an input of length n.

The exact running time of a Turing machine will often be a complex expression. To simplify things we only consider its asymptotic behaviour. This is done by only considering the highest order term of the expression and ignoring coefficients. We represent this with Big-O notation. This is best illustrated with an example.

Example 6.5. *If* $f(n) = 3n^7 + 3n^2 + 4$*, then* $f(n) = \mathcal{O}(n^7)$*.*

The following definition makes this more precise.

Definition 6.6. *Let* f, g *be two functions from* $\mathbb{N} \to \mathbb{R}$*. Then* $f(n) = \mathcal{O}(g(n))$ *if there exist* $c, n_0 \in \mathbb{N}$ *such that whenever* $n \geq n_0$*,* $f(n) \leq cg(n)$*.*

The above definition doesn't preclude a function f from having multiple representations in big-O notation. For instance if $f(n) = \mathcal{O}(n^7)$, then we also have that $f(n) = \mathcal{O}(n^k)$ whenever $k \geq 7$.

We say a language L can be decided in time $\mathcal{O}(f(n))$ if there exists a Turing machine M which decides L and has running time $t(n) = \mathcal{O}(f(n))$.

Turing machines can simulate all programming languages currently in use. The pseudocode used in this thesis can be simulated on a Turing machine. Furthermore if the runtime of an algorithm is bounded by a polynomial, then there exists a Turing machine which simulates that algorithm whose runtime is also bounded by a polynomial (though not necessarily the same one).

6.3 $\mathcal{P}$ and $\mathcal{NP}$

As mentioned, we think of languages as formalising the concept of a computational problem. Since we are concerned with solving computational problems efficiently, it is natural to consider which languages can be determined efficiently. A heuristic adopted by most mathematicians is to consider computational problems efficiently solvable if their solution can be determined in polynomial time. Phrased in terms of languages, we say a language L is decidable efficiently if it can be decided in time $\mathcal{O}(t(n))$ where $t(n)$ is a polynomial.

Definition 6.7. $\mathcal{P}$ *is the class of languages decidable in polynomial time.*

In *Section* 7 we will show that, when g is superadditive, the decision problem associated with determining the S-cast domination number of a graph lies in $\mathcal{P}$.

Not all languages lie in $\mathcal{P}$ and for many languages their membership is unknown. Of the languages where there is no known polynomial-time algorithm for determining them, some of them nonetheless have the property that they have *polynomial-time verifiable* solutions. This is best illustrated with an example.

Consider the dominating set problem. It's the language of strings which encode (G, k) where G has a dominating set of size k. There is no known polynomial-time algorithm for finding a dominating set in a graph G of size k. However if presented with a graph G and a set of k vertices which allegedly dominate G, one can efficiently check that indeed the set presented dominates G.

We define a verifier for a language L to be a Turing machine V such that $L = \{w \in \Sigma^* \mid V \text{ accepts } (w, c) \text{ for some } \Sigma\text{-string } c\}$. We call c the *certificate*. In the case of the dominating set problem, a certificate could be the string encoding the set of k dominating vertices.

Definition 6.8. *$\mathcal{NP}$ is the class of languages which have verifiers which halt in polynomial time.*

The biggest open problem in computational complexity is whether or not $\mathcal{P} = \mathcal{NP}$.

6.4 Polynomial Time Reductions and $\mathcal{NP}$-hardness

Certain problems may be solved by converting them into a new problem and then solving the new problem instead. Consider the problem of finding the largest clique in a graph G. A *clique* is a set $S \subseteq V(G)$ of vertices in a graph such that there is an edge connecting any two vertices in S. We can construct a new graph $\overline{G} = (V(G), \overline{E})$ where for vertices v and u, $\{v, u\} \in \overline{E}$ if and only if $\{v, u\} \notin E$ (this is called *the complement of* G). It is not hard to show that the largest clique in G has size k if and only if the largest independent set in $\overline{G}$ has size k. Thus the problem of finding the largest clique in G is solved just as easily by constructing $\overline{G}$ and finding the largest independent set.

Definition 6.9. *A function $f : \Sigma^* \to \Sigma^*$ is computable if there exists a Turing machine M such that on every input w, M halts with just $f(w)$ on the tape.*

If the Turing machine M in the above definition always halts in time polynomial we say f is *polynomial-time computable.*

Definition 6.10. *Let L and L' be languages. We say L is reducible to L' if there exists a computable function f such that $w \in L$ if and only if $f(w) \in L'$.*

If the computable function in the above definition is polynomial-time computable then we say that L is *polynomial-time reducible* to L'.

In [5] Stephen Cook showed that every problem in $\mathcal{NP}$ is polynomial-time reducible to the so-called boolean satisfiability problem which itself lies in $\mathcal{NP}$. This result was staggering at the time as it implied that $\mathcal{P}$ could be shown to equal $\mathcal{NP}$ simply by finding a polynomial-time solution to the boolean satisfiability problem. The following year Richard Karp showed in [13] that 21 other well known combinatorial problems in $\mathcal{NP}$ also share this property. This property is now known as $\mathcal{NP}$-completeness.

Definition 6.11. *A language $L \in \mathcal{NP}$ is $\mathcal{NP}$-complete if each language $L' \in \mathcal{NP}$ is polynomial-time reducible to L.*

If we remove the requirement that the language L belong to $\mathcal{NP}$ then we say that L is $\mathcal{NP}$-hard.

In order to show that a language L is $\mathcal{NP}$-hard, it is enough to find a polynomial-time reduction from an $\mathcal{NP}$-hard language L' to L. This is because for any language $X \in \mathcal{NP}$ there exists a polynomial-time computable function f which reduces X to L' and there also exists a polynomial-time reducible function f' which reduces L' to L. These function can be composed and the composed function runs in polynomial time.

6.5 Oracles

When analysing the complexity of certain problems we may not be interested in certain subroutines employed by Turing machines which solve them. We can treat certain subroutines as a black box and have a Turing machine query the black box for an immediate answer. This is a single motivation for the concept of an oracle machine, though further uses can be found in [20]. We use the definition given in [2].

An oracle machine is a Turing machine with an additional oracle tape, which the machine can write to and then query for an instantaneous answer. There are two types of queries we consider. If L is a language then an oracle can confirm whether or not a string w, belongs to the language L. In this case we write M^L to refer to such an oracle machine and call L the oracle. Alternatively if $f : \mathbb{N} \to \mathbb{N}$ is a function, then an oracle can return the function value $f(x)$ when queried with the string which encodes the number x. oracle machines of this form are denoted M^f and we call f an oracle.

New complexity classes can be formed by considering old complexity classes relative to an oracle. Complexity classes are often defined as all languages L which have a Turing machine M which decides them, satisfying some property (such as halting in polynomial time). When we consider a complexity class relative to an oracle L, we consider languages decided by oracle machines of the form M^L which intuitively satisfy the same property as the original class. With this in mind we define formally $\mathcal{P}$ taken relative to a functional oracle f.

Definition 6.12. *Let $f : \mathbb{N} \to \mathbb{N}$ be a function. Then $\mathcal{P}^f$ is the set of all languages decidable by oracle machines M^f which halt in polynomial time.*

Note that we require no restrictions on the form of the function f. In particular, it need not be computable.

Algorithms written in pseudo-code which use an oracle are simulatable by oracle machines and just as before the conversion doesn't significantly change the time complexity.

7 Algorithms

In this section we study the family of languages describing the S-cast domination problem.

Definition 7.1. *Given a scaling function g, let $S(g)$ be the language containing strings which encode (G, k) where $k \geq \gamma_s^g(G)$. We call $S(g)$ the S-cast domination problem on g.*

In [10] it is shown that the classical domination problem is $\mathcal{NP}$-hard. This problem can be reduced to the S-cast domination problem $S(g)$ where g is the scaling function given by $g(0) = 0$, $g(\infty) = \infty$ and $g(x) = 1$ for all other $x \in \mathbb{N}$. So $S(g)$ in this case in $\mathcal{NP}$-hard. In general we will not expect $S(g)$ to be $\mathcal{NP}$-hard for all g and in fact we show that when g is superadditive, $S(g) \in \mathcal{P}^g$ the complexity class of languages decided by oracle machines which use the function g as an oracle and halt in polynomial time. We adapt the argument presented in [12] which rely on exploiting structural properties of efficient dominating broadcasts.

In [14] it is shown that for a subadditive cost function k the cost-broadcast domination number $\gamma_c^k(G)$ can be calculated in polynomial time. We will prove and use this result to find $\gamma_s^g(G)$ for (G, g) assuming g is superadditive.

7.1 Graph Partitions by an Efficient Broadcast

Let f be an efficient dominating broadcast. Then the balls $B(v, f(v))$ for $v \in V_f^+$ partition the vertex set.

Definition 7.2. *Let (G, g) be a graph equipped with a scaling and cost function and let f be an efficient broadcast on (G, g). Let G_f be a new graph with vertex set V_f^+ and edge set equal to $\{uv \mid N(B(u, f(u))) \cap B(v, f(v)) \neq \emptyset\}$.*

Intuitively G_f is the graph obtained by considering each ball $B(v, f(v))$ for $v \in V_f^+$ as a vertex, with an edge between two balls $B(v, f(v))$ and $B(u, f(u))$ if a vertex in the one ball belongs to the neighbourhood of the other ball.

In addition to the usual domination parameters we now define a new one, $\gamma_{cp}^{g^*}(G)$, which is the minimum cost of an efficient dominating broadcast making G_f a path.

Lemma 7.3. *If f is an efficient broadcast on G and $v, u \in V_f^+$ then $d_G(v,u) \geq f(v) + f(u) + 1$.*

Proof. Assume $d_G(v,u) = k \leq f(v) + f(u)$. Then let P be a $v-u$ geodesic. Let $x \neq u$ be a vertex on P such that $d_G(v,x) = f(v)$. Such a vertex exists because f efficient implies f independent. Then $d_G(x,u) = k - f(v) \leq f(u)$. But this implies that $u \in B(u, f(u)) \cap B(v, f(v))$, which is a contradiction. $\square$

Lemma 7.4. *Let g be a superadditive scaling function and let f be an efficient dominating broadcast on (G,g). If G_f has a vertex with degree greater than 2 then there is an efficient broadcast f' with $|V_{f'}^+| < |V_f^+|$ and $w_c^{g^*}(f') = w_c^{g^*}(f)$.*

Proof. In the graph G_f, let v be a vertex with degree greater than 2 and let x, y, z be distinct neighbours of v. By the definition of G_f we know that $f(v), f(x), f(y), f(z)$ are all greater than 0 (when these vertices are treated as vertices of G). By Lemma 7.3 we know that $d_G(v,x) \geq f(v) + f(x) + 1$, but given that $N(B(u, f(u)) \cap B(v, f(v)) \neq \emptyset$ we see that $d_G(v,x) = f(v) + f(x) + 1$. This argument also gives us that $d_G(v,y) = f(v) + f(y) + 1$ and $d_G(x,z) = f(v) + f(z) + 1$. Without loss of generality assume that $f(x) \leq f(y) \leq f(z)$.

Assume $f(x) + f(y) > f(z)$. Then we construct a new broadcast f' with $f'(u) = f(u)$ for $u \in V - \{v, x, y, z\}$, $f'(v) = f(v) + f(x) + f(y) + f(z)$ and $f'(x) = f'(y) = f'(z) = 0$. This broadcast clearly has the same cost and a smaller positive vertex set. To see that it is still dominating consider a vertex u dominated by either x, y or z. Since $f(z) \geq f(y) \geq f(x)$ we have that $d_G(v,u) \leq f(v) + 2f(z) + 1$. By our assumption we get that $f'(v) \geq f(v) + 2f(z) + 1$ and so we are done. If f' is not efficient we apply the techniques used in Lemma 4.7 to find a broadcast f'' with the same cost as f' and $|V_{f''}^+| \leq |V_{f'}^+|$.

Next suppose $f(x) + f(y) \leq f(z)$. As mentioned $d_G(v,z) = f(v) + f(z) + 1$. So let P be a path from v to z of length $f(v) + f(z) + 1$. Let w be a vertex on P such that $d_G(w,z) = f(v) + f(x) + f(y)$. We know such a w exists because $f(x) + f(y) \leq f(z)$. Since f is efficient we conclude that $f(w) = 0$. Now we let $f'(u) = f(u)$ for all $u \in V - \{v, w, x, y, z\}$, $f(w) = f(v) + f(x) + f(y) + f(z)$ and $f(v) = f(x) = f(y) = f(z) = 0$. This broadcast has the same cost as f and fewer dominating vertices so it remains to check that f' is dominating. By construction, any vertex dominated by v or z with f is now dominated by

w with respect to f'. Assume u is a vertex dominated by y originally. We know that the distance from v to w is equal to $f(v)+f(z)+1-f(v)-f(x)-f(y) = f(z)-f(x)-f(y)+1$. Thus since $d_G(w,y) \leq d_G(w,v) + d_G(v,y)$ we conclude that $d_G(w,y) \leq (f(z) - f(x) - f(y) + 1) + (f(v) + f(y) + 1) = f(v) + f(z) + 2 - f(x) \leq f(v) + f(z) + f(x)$. Hence it follows that w f'-dominates u. A similar argument shows that any vertex dominated by x is now dominated by w. Thus f' is dominating and again, if it is not efficient, we can find a smaller function which is. $\square$

Theorem 7.5. *Let G be a graph. Then there exists an efficient optimal dominating broadcast f such that G_f is either a path or a cycle.*

Proof. Let f be an efficient optimal dominating broadcast where G_f is not a path or cycle. Then G_f has a vertex of degree greater than 2. Then we apply Lemma 7.4 to find a broadcast f' with fewer dominating vertices. We apply this process a number of times, each time reducing the size of the vertex set. Since it was finite to start this process must terminate and we are left with an optimal dominating broadcast f making G_f a path or cycle since G_f is always connected. $\square$

Corollary 7.6. *Let (G, g) be a graph. Then there exists an efficient optimal broadcast f such that removing the vertices of $B(v, f(v))$ from G results in at most two components, for each $v \in V_f^+$.*

Proof. We know by Theorem 7.5 that there is an efficient optimal broadcast f making G_f a path or a cycle. The vertices in G_f can be thought of as the balls $B(v, f(v))$ for $v \in V_f^+$. Thus since G_f is a path or cycle, removing any $B(v, f(v))$ will not split the graph into more than two components. $\square$

Lemma 7.7. *Let (G, g) be a graph G equipped with a scaling function g and let f be an efficient optimal broadcast. Then if $x \in V_f^+$, let G' be the subgraph of G induced by the vertex set $V' = V(G) - B(x, f(x))$. The broadcast f' on (G', g) defined by $f'(v) = f(v)$ for all $v \in V - B(x, f(x))$ is efficient and optimal.*

Proof. To see that f' is optimal assume there exists a broadcast t such that $w_c^{g^*}(t) < w_c^{g^*}(f)$. Then the broadcast t' defined on (G, g) with $t'(v) = t(v)$ for $v \in V'$, $t'(x) = f(x)$

and $t'(v) = 0$ for $v \in B(x, f(x)) - \{x\}$ is dominating and $w_c^{g^*}(t') < w_c^{g^*}(f)$, which is a contradiction. Furthermore, since f is efficient, we get immediately that f' is efficient. $\square$

Corollary 7.8. *Let (G, g) be a graph. Then there exists an efficient optimal dominating broadcast f on (G, g) and a vertex $x \in V_f^+$ such that the induced subgraph G', with vertex set equal to $V(G) - B(x, f(x))$, is either empty, or connected with $\gamma_c^{g^*}(G') = \gamma_{cp}^{g^*}(G')$.*

Proof. By Theorem 7.5 there exists an efficient optimal dominating broadcast f making G_f a path or a cycle. If G is the graph with one vertex then $G - B(x, f(x))$ is empty.

If G_f is a path of length greater than 1 then take $B(x, f(x))$ to be one of the endpoints. If G_f is a cycle take $B(x, f(x)$ equal to any vertex in G_f. In both cases $G_f - \{B(x, f(x))\}$ is a path. Now consider G' defined to be the subgraph of G induced by the vertex set $V' = V - B(x, f(x))$. Define f' on (G', g) by $f'(v) = f(v)$ for all $v \in V - B(x, f(x))$. By Lemma 7.7 f' is optimal and efficient. $G'_{f'}$ is clearly equal to $G_f - B(x, f(x))$, which we established was a path. Hence $\gamma_c^{g^*}(G') = \gamma_{cp}^{g^*}(G')$. $\square$

7.2 Finding an Optimal Broadcast When $\gamma_c^{g^*}(G) = \gamma_{cp}^{g^*}(G)$

We know that we can find an optimal dominating broadcast f such that f is efficient and G_f is a path or a cycle. First given a graph G we will give an algorithm that finds a broadcast f such that $w_c^{g^*}(f) \leq \gamma_{cp}^{g^*(G)}$. Then we will apply the above corollaries to find an algorithm that works in general.

The algorithm in this section is used to find a minimal dominating broadcast f making G_f a path with smallest cost. It works as follows: for each $u \in V(G)$ we construct a new digraph G_u and use this to find the best f_u making G_{f_u} a path and $u \in B(x, f_u(x))$ where $B(x, f_u(x))$ is one of the end vertices of G_{f_u}. We then select the best broadcast over all f_u's, which will have the desired property.

The vertices of G_u are the pairs (v, p) where $v \in V(G)$ and $p \in \{1, \ldots, \text{rad}(G)\}$ and

- $G(V(G) - B(v, p))$ is connected or empty and $u \in B(v, p)$,
- $G(V(G) - B(v, p))$ has at most two components and $u \notin B(v, p)$.

Thus there are at most $n \cdot \text{rad}(G)$ vertices in G_u.

Recall that G_u is a directed graph. Here u in some sense defines the left endpoint of the graph. To define the edge set we first partition the vertex set into four.

- $A_u = \{(v, p) \mid G(V(G) - B(v, p))$ is connected and $u \in B(v, p)\}$
- $B_u = \{(v, p) \mid G(V(G) - B(v, p))$ has two components$\}$
- $C_u = \{(v, p) \mid G(V(G) - B(v, p))$ is connected and $u \notin B(v, p)\}$
- $D_u = \{(v, p) \mid G(B(v, p)) = G\}$

For each vertex (v, p) define $L_u(v, p)$ to be the component of $G(V(G) - B(v, p))$ which contains u and and $R_u(v, p)$ the component of $G(V(G) - B(v, p))$ which does not contain u. Thus $L_u(v, p) = \emptyset$ for each $(v, p) \in A_u \cup D_u$ and $R_u(v, p) = \emptyset$ for each $(v, p) \in C_u$.

Now we define the arcs of G_u. There is an arc from (v, p) to (w, q) if and only if the following three conditions are satisfied:

- $B(v, p) \cap B(w, q) = \emptyset$ in G,
- $R_u(v, p) \neq \emptyset$ and $L_u(w, q) \neq \emptyset$,
- $(N_G(B(w, q)) \cap L_u(w, q)) \subseteq B(v, p)$ and $(N_G(B(v, p)) \cap R_u(v, p)) \subseteq B(w, q)$.

If there is an edge from (v, p) to (w, q) then it can be shown that $(N_G(B(w, q)) \cap L_u(w, q))$ is nonempty. As by the second condition $L_u(w, q) \neq \emptyset$ and since G is connected there are vertices adjacent to $B(w, q)$ in each component of $G(V(G) - B(w, q))$. Likewise it is the case that $(N_G(B(v, p)) \cap R_u(v, p)) \neq \emptyset$.

All vertices in A_u have in degree 0 and all vertices in C_u have out degree 0. Thus all paths in G_u can contain at most one vertex in A_u and at most one vertex in C_u. The vertices in D_u are isolated and define radial broadcasts in G.

In some sense the edges in G_u go from left to right. To make this idea rigorous consider the following lemma.

Lemma 7.9. *Every edge in G_u goes from left to right, which is to say that if there is an arc from (v,p) to (w,q) then $B(w,q) \subseteq R_u(v,p)$ and $B(v,p) \subseteq L_u(w,q)$.*

Proof. Since $\emptyset \neq (N_G(B(v,p)) \cap R_u(v,p)) \subseteq B(w,q)$, we get that $B(w,q) \cap R_u(v,p) \neq \emptyset$. Let $x \in B(w,q) \cap R_u(v,p)$ and assume $y \in B(w,q)$ and $y \notin R_u(v,p)$. Since $B(w,q)$ induces a connected subgraph of G we can find a path in G from x to y with each vertex on this path belonging to $B(w,q)$. Let b be the first vertex in this path not in $R_u(v,p)$. Because there are no edges between vertices in $R_u(v,p)$ to vertices in $L_u(v,p)$ it follows that b belongs to $B(v,p)$. But this contradicts the fact that $B(w,q) \cap B(v,p) = \emptyset$. The other direction uses an identical argument. $\square$

We now show that any path in G_u starting at a vertex in $A_u \cup D_u$ and ending in $B_u \cup C_u$ corresponds to a dominating broadcast.

Lemma 7.10. *Given a graph G and a vertex u in G. Let $P = (v_1,p_1), \ldots (v_k,p_k)$ be a path in G_u with $(v_1,p_1) \in A_u \cup D_u$ and $(v_k,p_k) \in B_u \cup C_u$. If P has length greater than 1 then $N_G(B(v_1,p_1)) \subseteq B(v_2,p_2)$, $N_G(B(v_k,p_k)) \subseteq B(v_{k-1},p_{k-1})$ and $N_G(B(v_i,p_i)) \subseteq B(v_{i-1},p_{i-1}) \cup B(v_{i+1},p_{i+1})$ for $1 < i < k$.*

Proof. Let $1 < i < k$ and $v \in N_G(B(v_i,p_i))$. Since $v \notin B(v_i,p_i)$, we have $v \in L_u(v_i,p_i)$ or $v \in R_u(v_i,p_i)$. Let $v \in R_u(v_i,p_i)$. Then $v \in R_u(v_i,p_i) \cap N_G(v_i,p_i) \subseteq B(v_{i+1},p_{i+1})$ since there is an arc from (v_i,p_i) to (v_{i+1},p_{i+1}). Similarly if $v \in L_u(v_i,p_i)$ we get that $v \in B(v_{i-1},p_{i+1})$.

In the case of $N_G(B(v_1,p_1))$ and $N_G(B(v_k,p_k))$, $L_u(v_1,p_1) = \emptyset$ and $R_u(v_k,p_k) = \emptyset$ and applying a similar argument we get the result required. $\square$

Lemma 7.11. *Let (v,p) and (w,q) be two vertices in G_u with an arc from (v,p) to (w,q). Then $L_u(v,p) \subsetneq L_u(w,q)$ and $R_u(w,q) \subsetneq R_u(v,p)$.*

Proof. If $L_u(v,p) = \emptyset$ then the first inclusion holds trivially.

Assume $L_u(v,p) \neq \emptyset$ and let $x \in L_u(v,p)$. If we can show that $x \notin B(w,q) \cup R_u(w,q)$ then it will follow that $x \in L_u(w,q)$. By Lemma 7.9 we have that $B(w,q) \subseteq R_u(v,p)$. Since $R_u(v,p) \cap L_u(v,p) = \emptyset$ we find that $x \notin B(w,q)$. If we assume that $x \in R_u(w,q)$

then, because $x \in L_u(v,p)$, we have a path P in $L_u(v,p)$ from u to x. We know in G_g that removing the vertices of $B(w,q)$ disconnects x from u. Hence a vertex $y \in B(w,q)$ must lie on P. We have already shown that $L_u(v,p) \cap B(w,q) = \emptyset$ and so this is a contradiction.

We have shown that $L_u(v,p) \subseteq L_u(w,q)$. To see that the sets are not equal observe that by Lemma 7.9, $B(v,p) \subseteq L_u(w,q)$ and that $B(v,p) \cap L_u(v,p) = \emptyset$.

A symmetric argument gives the proof for $R_u(w,q) \subsetneq R_u(v,p)$. □

Lemma 7.12. *$P = (v_1,p_1), \ldots (v_k,p_k)$ be a path in G_u. Then $v_i \neq v_j$ for all $i \neq j$.*

Proof. To begin with note that $v_i \neq v_{i+1}$. For if this were the case we would have $v_i \in B(v_i,p_i) \cap B(v_{i+1},p_{i+1})$ and so $B(v_i,p_i) \cap B(v_{i+1},p_{i+1}) \neq \emptyset$, contradicting the fact that there is an arc from (v_i,p_i) to (v_{i+1},p_{i+1}).

Assume that $v_i = v_{i+t}$ and that $t > 1$. Then by Lemma 7.11 we get chains $L_u(v_i,p_i) \subsetneq L_u(v_{i+1},p_{i+1}) \subsetneq \cdots \subsetneq L_u(v_{i+t},p_{i+t})$ and $R_u(v_{i+t},p_{i+t}) \subsetneq \cdots \subsetneq R_u(v_{i+1},p_{i+1}) \subsetneq R_u(v_i,p_i)$. If $p_i = p_{i+t}$ then $L_u(v_{i+1},p_{i+1}) = L_u(v_i,p_i)$, which is a contradiction. If $p_i < p_{i+t}$ then we would get $L_u(v_{i+t},p_{i+t}) \subsetneq L_u(v_i,p_i)$, which is a contradiction. Finally if $p_i > p_{i+1}$ we would get $R_u(v_i,p_i) \subsetneq R_u(v_{i+t},p_{i+t})$, which is also a contradiction. Hence no vertices $v_i = v_{i+t}$ exists in P. □

Now let $P = (v_1,p_1), \ldots (v_k,p_k)$ be a path in G_u with $(v_1,p_1) \in A_u \cup D_u$ and $(v_k,p_k) \in B_u \cup D_u$. Then let f_P be the broadcast defined by $f(v_i) = p_i$ if $(v_i,p_i) \in P$ and $f(v) = 0$ otherwise. Observe that f_P is well defined by Lemma 7.12.

Lemma 7.13. *Let $P = (v_1,p_1), \ldots (v_k,p_k)$ be a path in G_u with $(v_1,p_1) \in A_u \cup D_u$ and $(v_k,p_k) \in B_u \cup D_u$. Then f_P is a dominating broadcast on G_u.*

Proof. Let $P = (v_1,p_1), \ldots, (v_k,p_k)$ be a path in G_u with $(v_1,p_1) \in A_u \cup D_u$ and $(v_k,p_k) \in B_u \cup D_u$. Since the path begins in $A_u \cup D_u$ and ends in $B_u \cup C_u$ we know it has length greater than or equal to 2.

Let $S = \bigcup_i B(v_i,p_i)$. Assume $x \in V(G)$ and $x \notin S$. Since G is connected there is a path from x to v_1. Let z be the first vertex on this path which is contained in S. Let y be the vertex just before z on this path and let $z \in B(v_j,p_j)$. Then $y \in N_G(B(v_j,p_j))$ by

Lemma 7.10 we have then that y belongs to $B(v_2, p_2)$ if $j = 1$ or $B(v_{k-1}, p_{v-1})$ if $j = k$ or y belongs to $B(v_{j-1}, p_{j-1}) \cup B(v_{j+1}, p_{j+1})$ if $1 < j < k$. Either way, this contradicts the fact that y does not belong to S. Thus $S = V(P)$ and f_P is dominating. $\square$

If P is a path in G_u starting in $A_u \cup D_u$, then we say that the cost of P is equal to the cost of f_P.

We will now show that every efficient dominating broadcast f where G_f is a path corresponds to a path P in G_u for some vertex u in G.

Lemma 7.14. *Let f be an efficient dominating broadcast such that $G_f = u_1, u_2, \ldots, u_k$ is a path with $k \geq 3$. Then for each $1 < i < k$, $G(V(G) - B(u_i, f(u_i)))$ has two components $L_i = L_{u_1}(u_i, f(u_i))$ and $R_i = R_{u_1}(u_i, f(u_i))$.*

Proof. Since each $G(B(u_i, f(u_i)))$ is connected and there is adjacent edge between consecutive balls, we get that $\bigcup_{j=1}^{i-1} B(u_j, f(u_j))$ and $\bigcup_{j=i+1}^{k} B(u_j, f(u_j))$ are components and lie in $G(V(G) - B(u_i, f(u_i)))$. Furthermore there are no edges between each component as this would contradict the fact that G_f is a path.

Additionally, it is clear that $L_i = L_{u_1}(u_i, f(u_i)) = \bigcup_{j=1}^{i-1} B(u_j, f(u_j))$ and that $R_i = R_{u_1}(u_i, f(u_i)) = \bigcup_{j=i+1}^{k} B(u_j, f(u_j))$. $\square$

Lemma 7.15. *Let f be an efficient dominating broadcast such that $G_f = u_1, \ldots, u_k$ is a path. Then if $k = 1$ then $(u_1, f(u_1) \in D_{u_1}$. If $k \geq 2$ then $(u_1, f(u_1)) \in A_{u_1}$, $(u_i, f(u_i)) \in B_{u_1}$ for $1 < i < k$ and $(u_k, f(u_k)) \in C_{u_1}$.*

Proof. If $k = 1$ then f is a radial broadcast and $B(u_1, f(u_1)) = V(G)$. Consequently $(u_1, f(u_1)) \in D_{u_1}$.

Assume $k \geq 2$. To show that $(u_1, f(u_1)) \in A_{u_1}$ we must show that that $u_1 \in B(u_1, f(u_1))$ which is clearly true and that $G(V(G) - B(u_1, f(u_1)))$ is connected. This follows because each $B(u_i, f(u_i))$ is connected and there is an edge between consecutive balls (since G_u is a path).

A similar argument shows that $G(V(G) - B(u_k, f(u_k)))$ is connected and since each pair of balls are disjoint we have that $u_1 \notin B(u_k, f(u_k))$. It follows that $(u_k, f(u_k)) \in C_{u_1}$.

By Lemma 7.14 we see that for $1 < i < k$, $G(V(G) - B(u_i, f(u_i)))$ has two components, which gives that $(u_i, f(u_i)) \in B_{u_1}$. $\square$

Lemma 7.16. *Let f be an efficient dominating broadcast on G such that $G_f = u_1, \ldots, u_k$ is a path. In G_{u_1} there is an edge between $(u_i, f(u_i))$ and $(u_{i+1}, f(u_{i+1}))$, for $1 \leq i < k$.*

Proof. There are three conditions that must be satisfied in order for an edge to lie between $(u_i, f(u_i))$ and $(u_{i+1}, f(u_{i+1}))$. The first is that $B(u_i, f(u_i)) \cap B(u_{i+1}, f(u_{i+1})) = \emptyset$, which is true because of the efficiency of f.

The second condition requires that both $R_{u_1}(u_i, f(u_i)) \neq \emptyset$ and $L_{u_1}(u_{i+1}, f(u_{i+1})) \neq \emptyset$. Lemma 7.14 gives us that $R_{u_1}(u_i, f(u_i)) = R_i = \cup_{j=i+1}^{k} B(u_j, f(u_j))$, which is not empty because $i < k$. In addition it gives $L_{u_1}(u_{i+1}, f(u_{i+1})) = L_i = \cup_{j=1}^{i-1} B(u_j, f(u_j))$, which is not empty because $i + 1 > 1$.

The final condition requires that $N_G(B(u_{i+1}, f(u_{i+1}))) \cap L_i \subseteq B(u_i, f(u_i))$ and that $N_G(B(u_i, f(u_i)) \cap R_i \subseteq B(u_{i+1}, f(u_{i+1}))$. For $s > i + 1$, we find that $N_G(B(u_i, f(u_i))) \cap B(u_s, f(u_s)) = \emptyset$ as there are no edges between u_i and u_s in G_f. Thus $N_g(B(u_i, f(u_i))) \cap R_i \subseteq B(u_{i+1}, f(u_{i+1}))$. An identical argument gives that $N_G(B(u_{i+1}, f(u_{i+1}))) \cap L_i \subseteq B(u_i, f(u_i))$. $\square$

Corollary 7.17. *Let f be an efficient dominating broadcast on G with $G_f = u_1, \ldots, u_k$ a path. Then $(u_1, f(u_1)), \ldots, (u_k, f(u_k))$ is a directed path in G_{u_1} beginning in $A_{u_1} \cup D_{u_1}$ and ending in $C_{u_1} \cup D_{u_1}$.*

Proof. By Lemma 7.15 we know that $(u_1, f(u_1)) \in A_{u_1} \cup D_{u_1}$, $(u_i, f(u_i)) \in B_{u_1}$ for $1 < i < k$ and $(u_k, f(u_k)) \in C_{u_1} \cup D_{u_1}$. Furthermore, we know there is an edge between each consecutive pair of edges by Lemma 7.16. $\square$

Now we consider an algorithm which finds the minimum cost of a broadcast on G with subadditive cost function g^* when $\gamma_c^{g^*}(G) = \gamma_{cp}^{g^*}(G)$. It uses an oracle which returns the value of $g^*(n)$ given input $n \in \mathbb{N}$.

Data: A graph G

Result: An efficient dominating broadcast f with $w_c^{g^*}(f) \leq \gamma_{cp}^{g^*}(G)$

begin

  for *each vertex* $v \in V(G)$ **do**

    $f(v) = 0$

  end

  Let P be a dummy path costing $g^*(\mathrm{rad}(G)) + 1$

  for *each vertex* $u \in V(G)$ **do**

    Compute G_u, A_u, B_u, C_u and D_u

    Find a minimum cost path P_u beginning in $A_u \cup D_u$ and ending in $C_u \cup D_u$

    if P_u *costs less than* P **then**

      Set $P = P_u$

    end

  end

  for *each vertex* $(v, p) \in P$ **do**

    $f(v) = p$

  end

end

Algorithm 1: Adapted Minimum Cost Path Broadcast Algorithm With Respect to g^* — AMDBP^{g^*}

Theorem 7.18. *Given a graph G and subadditive cost function g^*,* AMDBP^{g^*} *returns a broadcast such that $w_c^{g^*}(f) \leq \gamma_{cp}^{g^*}(G)$.*

Proof. Let f' be a dominating broadcast with $G_{f'} = u_1, \ldots, u_k$ a path and $w_c^{g^*}(f') = \gamma_c^{g^*}(G)$. Then by Corollary 7.17, $G_{f'}$ corresponds to a path in G_{u_1}. AMDBP finds a minimum weighted path P from $A_u \cup D_u$ to $C_u \cup D_u$ over all $u \in V(G)$. By Lemma 7.13 we know that the broadcast returned is f_P which is dominating and we have that $w_c^{g^*}(f_P) \leq w_c^{g^*}(f') = \gamma_{cg}^{g^*}(G)$. □

Corollary 7.19. *Let g^* be a subadditive cost function and G is a graph with $\gamma_{cp}^{g^*}(G) = \gamma_c^{g^*}(G)$. Then* AMDBP^{g^*} *returns an optimal efficient dominating broadcast on G with respect to g^*.*

Theorem 7.20. *Given a graph G with n vertices and subadditive cost function g^*,* AMDBP^{g^*} *runs in $\mathcal{O}(n^7)$.*

Proof. The algorithm computes the radius of G which can be done in n^3 time.

The first `for` loop initialises the broadcast f and takes n steps.

In the major `for` loop, G_u is computed. G_u has $\mathcal{O}(n^2)$ vertices and consequently has up to n^4 edges. Each pair of vertices should have the three edge properties checked.

$B(v,p) \cap B(w,q) = \emptyset$ can be determined in n^2 time using a breadth first search.

$R_u(v,p) \neq \emptyset$ and $L_u(w,q) \neq \emptyset$ can also be determined with a breadth first search, taking at most n^2 time.

$N_G(B(w,q) \cap L_u(w,q)) \subseteq B(v,p)$ and $N_G(B(v,p) \cap R_u(v,p)) \subseteq B(w,q)$ can also be determined in n^2 time with a breadth first search.

Finding A_u, B_u, C_u and D_u can be done by determining if a graph is connected which can be done with a breadth first search in n^2 time.

Finding the minimum path on G_u can be done in $n^4 \log(n)$ time using a modified Dijkstra's algorithm.

Thus the algorithm terminates in time $\mathcal{O}(n + n^3 + n(n^6 + n^4 \log(n))) = \mathcal{O}(n^7)$. $\square$

For details on the breadth first search and Dijkstra's algorithm see [6].

7.3 Broadcast Domination For All Cases

To compute an optimal broadcast domination in the general case we consider $G' = G(V(G) - B(x,k))$ for each $x \in V(G)$ and $k \in \underline{\mathrm{rad}(G)}$ and when G' is connected or empty run AMDPB$^{g^*}$ on G', extending its output to G. The algorithm presented also uses g^* as an oracle.

Data: A graph G

Result: An optimal dominating broadcast f

```
begin
    cost = rad(G) + 1
    for each vertex v ∈ V(G) do
        f(v) = 0
    end
    for each vertex x in V(G) do
        for each k in rad(G) do
            if G' = G(V(G) − B(x,k)) is connected or empty then
                f' = AMDPB^{g*}(G')
                if w_c^{g*}(f') + g*(k) < cost then
                    cost = w_c^{g*}(f') + g*(k)
                    for each v ∈ V(G') do
                        f(v) = f'(v)
                    end
                    f(x) = k
                    for v ∈ B(x,k) − {x} do
                        f(v) = 0
                    end
                end
            end
        end
    end
end
```

Algorithm 2: Adapted Optimal Broadcast Domination on g^* — AOBD^{g^*}

Theorem 7.21. AOBD^{g^*} *computes an efficient optimal dominating broadcast on* (G, g).

Proof. Corollary 7.8 says that there exists an efficient optimal broadcast dominating function f and a vertex $x \in V(G)$ such that $G' = G(V(G) - B(x, f(x)))$ is either empty or connected and that $\gamma_c^{g^*}(G') = \gamma_{cp}^{g^*}(G')$. Furthermore Lemma 7.7 allows us to extend an optimal broadcast on G' to an efficient optimal broadcast on G as performed in AOBD.

Thus since AOBD^{g^*} considers each graph $G' = G - B(v, x)$ we conclude that it returns an efficient optimal dominating broadcast. $\square$

Theorem 7.22. AOBD^{g^*} *runs in time* $\mathcal{O}(n^9)$*.*

Proof. The algorithm computes the radius of the graph which can be done in n^3 time. The inner loop checks for connectivity which can be checked using a breadth first search in n^2 time. It also computes AMDPB^{g^*} which takes n^7 time. The inner loop is performed n^2 times so the total time for the algorithm is n^9. □

Corollary 7.23. *If* g *is a superadditive scaling function then* $S(g) \in \mathcal{P}^g$*.*

Proof. Consider an oracle machine M^g which takes in a string as input and runs AOBD^{g^*} on G. If the string does not encode something of the form (G, k) where G is a graph and $k \in \mathbb{N}$ then the Turing machine immediately rejects. If the string is of the right form then it runs AOBD^{g^*} and checks if $k \geq \gamma_c^{g^*}(G)$, accepting if this is true and rejecting otherwise.

The oracle machine M^g halts in polynomial time since AOBD^{g^*} runs in polynomial time. By *Theorem* 3.51, $\gamma_s^g(G) = \gamma_c^{g^*}(G)$. Thus since AOBD^{g^*} finds $\gamma_c^{g^*}(G)$ the oracle machine can correctly determine if $k \geq \gamma_s^g(G)$. □

We use g as an oracle in order to avoid considering the runtime of computing g (or whether or not g is even computable). However if g^* is polynomial-time computable then the above shows that $S(g) \in \mathcal{P}$.

7.4 Scaling Functions Making the S-cast Domination Problem $\mathcal{NP}$-hard

In order to define the subadditive scaling functions, we define two additional functions.

Let $q : \mathbb{Z} \to \mathbb{N}$ with $q(x) = 3 \cdot 2^{x-1}$ for $x \geq 1$, $q(0) = q(-1) = 1$ and $q(x) = 0$ for $x \leq -2$. Then define the function $t : \mathbb{N} \to \mathbb{N}$ with $t(x) = \sum_{i \in \mathbb{N}} q(x - 3i)$.

Theorem 7.24. *If* g *is a scaling function with* $g(1) = 1$ *and* $g(t(x) + 1) < x$*, then* $S(g)$ *is* $\mathcal{NP}$*-hard.*

Since t is an exponential function it follows that the theorem applies when g grows at most logarithmically.

In [10] it is shown that the broadcast domination problem on graphs with maximum degree less than or equal to 3 is $\mathcal{NP}$-hard. We will show that when G is a graph whose maximum degree is less than or equal to 3 and g a scaling function with $g(1) = 1$ and $g(t(x) + 1) < x$ then $\gamma(G) = \gamma_s^g(G)$.

Lemma 7.25. *Let G be a graph with maximum degree less than or equal to 3 and g a scaling function satisfying the conditions in Theorem 7.24. Then any S-cast h on (G, g) with $h(v) > 1$ for some vertex $v \in V(G)$ is not optimal.*

Proof. Recall $q : \mathbb{Z} \to \mathbb{N}$ with $q(x) = 3 \cdot 2^{x-1}$ for $x \geq 1$, $q(0) = q(-1) = 1$ and $q(x) = 0$ for $x \leq -2$. Let $t : \mathbb{N} \to \mathbb{N}$ with $t(x) = \sum_{i \in \mathbb{N}} q(x - 3i)$. We have $g(1) = 1$ and $g(t(x)+1) < x$.

Let h be an S-cast on (G, g) with $h(v) = d > 1$, for some vertex $v \in V(G)$. Let $w \in N_h[v]$ with $d_G(w, v) = c = (g \circ h)(v)$. By our assumptions of g we have that $g(t(c) + 1) < c$. Thus $h(v) > t(c) + 1$.

We now construct subsets of $V(G)$. Let $A_{-2} = A_{-1} = A_0 = \{v\}$. Let A_1 be the set of vertices u where u is adjacent to v. Since G has maximum degree less than or equal to 3, we know $|A_1| \leq 3$. Let A_i for $i \geq 2$ be the set of vertices which are distance i from v. Since each vertex in A_i has maximum degree 3 and is adjacent to a vertex in A_{i-1} we get that $|A_i| \leq 3 \cdot 2^{i-1} = q(i)$ (when $i \geq 1$). We have that v h-dominates all vertices in each A_i for $1 \leq i \leq c$ since a vertex in A_i is at most distance i from v.

Construct a new S-cast h' with $h'(x) = h(x) + 1$ if $x \in (A_c \cup A_{c-3} \cup \cdots \cup A_{c-3l}) - \{v_k\}$ where $c - 3l \in \{0, -1, -2\}$ and $h'(v) = 1$ if $c - 3l \in \{-1, -2\}$ and $h'(v) = 0$ if $c - 3l = 0$. Finally $h'(x) = h(x)$ for all other vertices.

The number of vertices in $A_c \cup A_{c-3} \cup \cdots \cup A_{c-3l}$ is less than or equal to $t(c)$. Since we have added one to fewer than $t(c)$ vertices and reduced $h(v)$ to 0 or 1 where it was greater than $t(c) + 1$ before, we get that $w_s^{g^*}(h') < w_s^{g^*}(h)$. We must just check that h' is dominating. Let $u \in N_h(v)$. Then $u \in A_i$ for some $1 \leq i \leq c$. Either A_{i-1}, or A_i, or A_{i+1} is a subset of $V_{h'}^+$. In each case we see that u is dominated. This is a contradiction and so every optimal dominating S-cast h must have $h(v) \leq 1$. $\square$

Proposition 7.26. *If G is a graph with maximum degree less than or equal to 3 and g a scaling function satisfying the conditions in Theorem 7.24, then $\gamma_s^g(G) = \gamma(G)$.*

Proof. Let h be an optimal dominating broadcast on (G, g). Then by Lemma 7.25 we know that $h(v) \leq 1$ for any $v \in V(G)$. Consequently $(g \circ h)(v) \leq 1$. Thus if S is an optimal dominating set on G, then h_S is a dominating S-cast on G. We have that $w_s^g(h_s) = |S|$ and so $\gamma(G) \leq \gamma_s^g(G)$. Furthermore any optimal dominating S-cast on G is a characteristic function and consequently corresponds to a dominating set. Thus $\gamma_s^g(G) \leq \gamma(G)$.

$\square$

Thus we have a reduction from an $\mathcal{NP}$-hard problem to $S(g)$ when g is a scaling function with $g(1) = k \neq 0$ and $g(t(x) + 1) < x \cdot k$. Thus $S(g)$ is $\mathcal{NP}$-hard when g is of the form described above.

8 S-cast Domination of Products of Graphs

A *graph product* is a way to form a new graph from two graphs G and H. It has vertex set $V(G) \times V(H)$ and edge set depending on the product being considered. We consider three graph products in this thesis, the Cartesian product, the Tensor Product and the strong product. The Cartesian and strong product were introduced by Sabidussi in [18]. According to [15], the tensor product was first studied by Russell and Whitehead in [17] as a product on binary relations.

In this section we generalise some of the results found in [4] to the S-cast setting. If G and H are graphs and g a scaling function, then for each of the three products above we aim to bound the S-cast domination of the product of G and H in terms of a linear combination of $\gamma_s^g(G)$ and $\gamma_s^g(H)$.

8.1 S-cast Domination Number and the Radius

Definition 8.1. *A subgraph T of G is a spanning tree if T is a tree and $V(G) = V(T)$.*

Lemma 8.2. *Let G be a graph, H a spanning tree of G and g a scaling function. Then $\gamma_s^g(G) \leq \gamma_s^g(H)$ and $\mathrm{rad}(G) \leq \mathrm{rad}(H)$.*

Proof. G can be obtained from H by adding edges. Consequently $d_G(x, y) \leq d_H(x, y)$. From this it follows that any dominating broadcast on H is a dominating broadcast on G and also that $\mathrm{rad}(G) \leq \mathrm{rad}(H)$. $\square$

Lemma 8.3. *Let G be a graph and g a superadditive scaling function. Then there exists a spanning tree T of G such that $\gamma_s^g(T) = \gamma_s^g(G)$.*

Proof. Since g is superadditive there exists an efficient minimal dominating S-cast h on (G, g). Thus the neighbourhoods $N_h[v]$ for $v \in V_f^+$ are all pairwise disjoint. Consider the subgraphs generated by $N_h[v]$ and for each finding a spanning tree T_v such that $d_G(v, x) = d_{T_v}(v, x)$ for each $x \in N_h[v]$. The trees T_v can be glued together by identifying vertices in particular trees, in such a way that what results is a tree T, in particular a spanning tree of G. Furthermore h dominates T by construction and so $\gamma_s^g(T) \leq \gamma_s^g(G)$. This taken with Lemma 8.2 gives that $\gamma_s^g(T) = \gamma_s^g(G)$. $\square$

Lemma 8.4. *Let $G = P_n$ and g be a subadditive scaling function. Then $\gamma_s^g(P_n) = \lceil \frac{n}{2g(1)+1} \rceil$.*

Proof. Note that if g is subadditive and nontrivial then $g(1) > 0$. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of P_n with an edge between v_i and v_j if and only if $|i-j| = 1$. First we show that there exists an optimal dominating S-cast h on (P_n, g) such that $h(v_i) \leq 1$, for any vertex $v_i \in V_h^+$. Assume that h is a dominating broadcast and $h(v_i) > 1$ for some $v_i \in V_h^+$. Then $(g \circ h)(v_i) > g(1)$. Without loss of generality we may assume that $d_{P_n}(v_i, v_1) \geq (g \circ h)(v_i)$ and $d_{P_n}(v_n, v_i) \geq (g \circ h)(v_i)$, since if this didn't hold for a vertex v_i we could consider a new S-cast $\widetilde{h}$ which is the same as h except that $\widetilde{h}(v_i) = 0$ and $\widetilde{h}(v_{(g\circ h)(v_i)+1}) = h(v_i)$. Then $\widetilde{h}$ would be optimal dominating and have this desired property.

The set $N_h[v_i]$ is given by

$$N_h[v_i] = \{v_{i-g(h(v_i))}, v_{i-g(h(v_i))+1}, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{i+g(h(v_i)-1}, v_{i+g(h(v_i))}\}.$$

Then set

$$\begin{aligned} x &= i - g(h(v_i)) + g(\lfloor h(v_i)/2 \rfloor), \\ y &= i - g(h(v_i)) + 2g(\lfloor h(v_i)/2 \rfloor) + g(\lceil h(v_i)/2 \rceil) + 1, \end{aligned}$$

and consider a new broadcast

$$h'(v_j) = \begin{cases} \lfloor h(v_i)/2 \rfloor & j = x \\ 0 & j = i \\ \lceil h(v_i)/2 \rceil & j = y \\ h(v_j) & j \notin \{i, x, y\}. \end{cases}$$

To see that h' is dominating consider a vertex $v_k \in N_h[v_i]$. If $i - g(h(v_i)) \leq k \leq x + g(\lfloor h(v_i)/2 \rfloor)$ then it is easy to verify that $d_{P_n}(v_k, v_x) \leq g(\lfloor h(v_i)/2 \rfloor)$ and so $v_k \in N_h[v_x]$.

On the other hand take $x + g(\lfloor h(v_i)/2 \rfloor) + 1 \leq k \leq i + g(h(v_i))$. To bound $d_{P_n}(v_k, v_y)$ it is enough to consider the two boundary cases. For $k = x + g(\lfloor h(v_i)/2 \rfloor) + 1$ we have

$$\begin{aligned}
d_{P_n}(v_k, v_y) &= y - k \\
&= (i - g(h(v_i)) + 2g(\lfloor h(v_i)/2 \rfloor) + g(\lceil h(v_i)/2 \rceil) + 1) \\
&\quad - (i - g(h(v_i)) + 2g(\lfloor h(v_i)/2 \rfloor) + 1) \\
&= g(\lceil h(v_i)/2 \rceil)).
\end{aligned}$$

When $k = i + g(h(v_i))$ we have

$$\begin{aligned}
d_{P_n}(v_y, v_k) &= k - y \\
&= i + g(h(v_i)) - (i - g(h(v_i)) + 2g(\lfloor h(v_i)/2 \rfloor) + g(\lceil h(v_i)/2 \rceil) + 1) \\
&= 2g(h(v_i)) - 2g(\lfloor h(v_i)/2 \rfloor) - g(\lceil h(v_i)/2 \rceil) - 1 \\
&= 2(g(h(v_i)) - g(\lfloor h(v_i)/2 \rfloor)) - g(\lceil h(v_i)/2 \rceil) - 1 \\
&\leq 2g(\lceil h(v_i)/2 \rceil) - g(\lceil h(v_i)/2 \rceil) - 1 \qquad (1) \\
&\leq g(\lceil h(v_i)/2 \rceil),
\end{aligned}$$

where (1) follows from from the subadditivity of g, as we can show $g(h(v_i)) - g(\lfloor h(v_i)/2 \rfloor) \leq g(\lceil h(v_i)/2 \rceil)$. Thus $v_k \in N_{h'}[v_y]$ and so h' is dominating. Furthermore $w_s^g(h') = w_s^g(h)$. We can repeat the above procedure to produce a sequence of S-casts $h, h', h'', \ldots$ until we obtain an S-cast $h^{(t)}$ such that $h^{(t)}(v) \leq 1$ for any vertex $v \in V(G)$.

Thus there is an optimal dominating broadcast h of this form. Then $|N_h(v)| = 2g(1) + 1$ for each vertex $v \in V_h^+$. Thus we can conclude that $w_s^g(h) = \lceil \frac{n}{2g(1)+1} \rceil$ and consequently that $\gamma_s^g(P_n) = \lceil \frac{n}{2g(1)+1} \rceil$. □

This result is a generalisation of the broadcast result stating $\gamma_b(P_n) = \lceil \frac{n}{3} \rceil$.

Lemma 8.5. *Let T be a tree and g a subadditive scaling function. Then*

$$\gamma_s^g(T) \geq \left\lceil \frac{2\,\mathrm{rad}(T)}{2g(1) + 1} \right\rceil.$$

Proof. If $T = P_2$ then the bound clearly holds. Let $T \neq P_2$ be a tree and let $v \in V(T)$ be a vertex in the centre of T. Then there exists a vertex $u \in V(T)$ such that $d_T(v, u) = \mathrm{rad}(T)$. There exists a vertex $y \in V(T) - \{v\}$ in a component of $T(V(T) - \{v\})$ not containing u such that $d_T(y, v) \geq \mathrm{rad}(T) - 1$. For assume no such vertex y exists then consider a $v-u$ geodesic P and let $x \in P$ be the vertex adjacent to v. Then $d_T(x,u) = \operatorname{rad}(T) - 1$ and if z belongs to the same component as u in $T(V(T) - \{v\})$ then $d_T(v,z) \leq \operatorname{rad}(T) - 1$. If z belongs to a component not containing u then by our assumption we get that $d_T(x,z) \leq \operatorname{rad}(T) - 1$. This means that $\operatorname{rad}(T) = \operatorname{rad}(T) - 1$, which is a contradiction. Now let w be a vertex in a component of $V(V(T) - \{v\})$ such that $d_T(v,w) = \operatorname{rad}(T) - 1$. Then $d_T(w,u) = 2\operatorname{rad}(T) - 1$. Let P be a $w-u$ geodesic and consider the subgraph generated by $V(P)$. This subgraph is isomorphic to $P_{2\operatorname{rad}(T)}$. For each $x \in V(T)$ denote by $\widetilde{x}$ the unique closest vertex to x which lies on P.

Let h be an optimal dominating broadcast on T. Then define an S-cast h' on $T(V(P))$ with $h'(x) = \max\{h(y) \mid \widetilde{y} = x\}$. To see that h' is dominating consider a vertex $x \in V(P)$. Since h is dominating there exists a vertex $t \in V_h^+$ such that $x \in N_h[t]$. Then $d_T(x,t) \leq (g \circ h)(t)$. Since T is a tree there is a unique $t-x$ path of length less than or equal to $(g \circ h)(t)$. Starting from t, let t' be the first vertex of P to lie on this $t-x$ path. It is not hard to see that in fact $t' = \widetilde{t}$. Clearly $d_T(\widetilde{t}, x) \leq (g \circ h)(t)$ and from the construction of h', we find $(g \circ h)(t') = (g \circ h')(\widetilde{t}) \geq (g \circ h)(t)$.

Furthermore it is clear that $\gamma_s^g(T) = w_s^g(h) \geq \sum_{v \in V(P)} h(v) \geq \gamma_s^g(P_{2\operatorname{rad}(T)}) = \lceil \frac{n}{2g(1)+1} \rceil$. □

Corollary 8.6. *Let G be a graph and g a linear scaling function. Then $\gamma_s^g(G) \geq \lceil \frac{2\operatorname{rad}(T)}{2g(1)+1} \rceil$.*

Proof. Linear functions are precisely the functions which are super- and subadditive. When g is superadditive there exists a spanning tree T of G such that $\gamma_s^g(G) = \gamma_s^g(T) \geq \lceil \frac{2\operatorname{rad}(T)}{2g(1)+1} \rceil$ (by Lemma 8.5). □

Rearranging the above inequality, we get $\operatorname{rad}(G) \leq \frac{2g(1)+1}{2} \gamma_s^g(G)$.

8.2 The Cartesian Product

Definition 8.7. *Let G and H be two graphs. Then the Cartesian product of G and H, denoted $G \,\square\, H$, is the graph with vertex set $V(G \,\square\, H) = V(G) \times V(H)$, the Cartesian product of $V(G)$ and $V(H)$, and an edge between (u,u') and (v,v') if*

1. *$u = v$ and $\{u', v'\} \in E(H)$ or,*

2. $u' = v'$ and $\{u, v\} \in E(G)$.

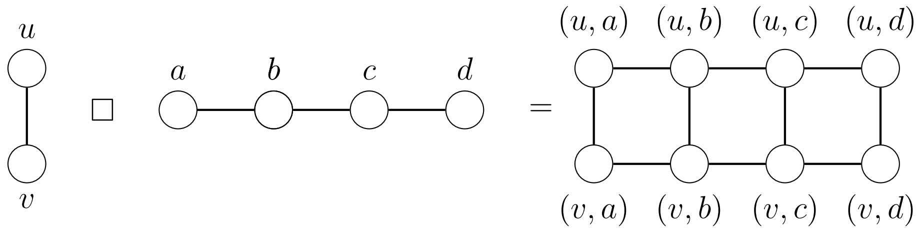

Figure 11: $P_2 \square P_4$

Proposition 8.8. *Consider connected graphs G and H and $G \square H$ the Cartesian product of G and H. Then*

$$d_{G\square H}((u, u'), (v, v')) \leq d_G(u, v) + d_H(u', v').$$

Proof. Consider two vertices $(u, u'), (v, v') \in V(G \square H)$. Since G and H are connected there exists geodesics $P = u, uw_2, w_2, \ldots, w_{k-1}v, v$ and $Q = u', u'w_2', w_2', \ldots, w_{l-1}'v', v'$. Then there is a path $Q' = (u, u'), (u, u')(u, w_2'), (u, w_2'), \ldots, (u, w_{l-1}')(u, v'), (u, v')$ of length $d_H(u', v')$ and path $P' = (u, v'), (u, v')(w_2, v'), (w_2, v'), \ldots, (w_k, v')(v, v'), (v, v')$ of length $d_G(u, v)$. These paths can be glued together to form a path from (u, v) to (u', v') of length $d_G(u, v) + d_H(u', v')$.

□

Corollary 8.9. *Let G and H be connected graphs. Then* $\mathrm{rad}(G \square H) \leq \mathrm{rad}(G) + \mathrm{rad}(H)$.

Corollary 8.10. *If G and H are connected graphs, then $G \square H$ is connected.*

Proposition 8.11. *Let G and H be graphs, let g be a scaling function and suppose one of the following two conditions holds:*

1. *G and H are trees and g is subadditive,*
2. *g is linear.*

Then we have

$$\gamma_s^g(G \square H) \leq g^*(\frac{2g(1) + 1}{2}(\gamma_s^g(H) + \gamma_s^g(H))).$$

Proof. By Corollary 8.9 we have that $\gamma_s^g(G \,\square\, H) \leq g^*(\mathrm{rad}(G \,\square\, H)) \leq g^*(\mathrm{rad}(G) + \mathrm{rad}(H))$. If condition 1 holds, we can apply Lemma 8.5 (or if condition 2 holds, apply Corollary 8.6) to get $g^*(\mathrm{rad}(G)+\mathrm{rad}(H)) \leq g^*(\frac{2g(1)+1}{2}(\gamma_s^g(G)+\gamma_s^g(H)))$. Combining these we find that $\gamma(G \,\square\, H) \leq g^*(\frac{2g(1)+1}{2}(\gamma_s^g(G) + \gamma_s^g(H)))$. □

In the case of standard broadcast domination we have

$$\gamma_b(G \,\square\, H) \leq \frac{3}{2}(\gamma_b(H) + \gamma_b(H)).$$

8.3 The Tensor Product

Definition 8.12. *Let G and H be two graphs. Then the tensor product of G and H, denoted $G \times H$, is the graph with vertex set $V(G \times H) = V(G) \times V(H)$, the Cartesian product of $V(G)$ and $V(H)$ and an edge between (u, u') and (v, v') if $\{u, v\} \in E(G)$ and $\{u', v'\} \in E(H)$.*

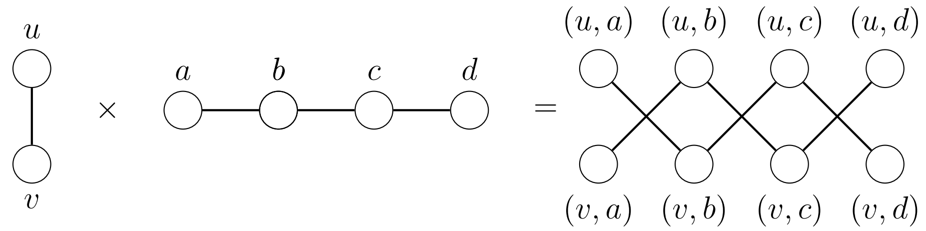

Figure 12: $P_2 \times P_4$

In [1] distances in $G \times H$ is studied extensively. We reproduce some of the results in this section.

Lemma 8.13. *Let G and H be connected graphs and let $u, v \in V(G)$ and $u', v' \in H$ with $W = w_1, w_1w_2, w_2, \ldots, w_{k-1}w_k, w_k$ a $u - v$ walk in G and $W' = w'_1, w'_1w'_2, w'_2, \ldots, w'_{k-1}w'_k, w'_k$ a $u' - v'$ walk in H. Then $W \times W' = (w_1, w'_1), (w_1, w'_1)(w_2, w'_2), (w_2, w'_2), \ldots, (w_k, w'_k)$ is a $(u, u') - (v, v')$ walk in $G \times H$.*

Proof. We know that in W, w_i is adjacent to w_{i+1} and from W' that w'_i is adjacent to w'_{i+1}. Hence $(w_i, w'_i)(w_{i+1}, w'_{i+1})$ is an edge between (w_i, w'_i) and (w_{i+1}, w'_{i+1}) which means that what is described in the lemma is a walk. □

Both walks in the lemma above are the same length. It is clear that $|W \times W'| = |W| = |W'|$.

Lemma 8.14. *Let G and H be connected graphs and let W be a $u-v$ walk in G and W' a $u'-v'$ walk in H. If the lengths of W and W' have the same parity, then $d((u,u'),(v,v')) \leq \max\{|W|, |W'|\}$.*

Proof. If $|W| = |W'|$ then we construct the $(u,u')-(v,v')$ walk $W \times W'$ of length equal to $|W| = |W'|$ and so the inequality is satisfied.

Assume $|W| < |W'|$ and let $W = u, uw_2, w_2, \ldots, w_{k-1}, w_{k-1}v, v$. Since $|W|$ and $|W'|$ have the same parity, W can be extended to a walk $\overline{W}$ with $|\overline{W}| = |W'|$ by appending $vw_{k-1}, w_{k-1}, w_{k-1}v, v$ to the end of W as many times as necessary. Then $\overline{W} \times W'$ is a $(u,u')-(v,v')$ walk of length $|W'|$ which gives $d_{G\times H}((u,u'),(v,v')) \leq \max\{|W|, |W'|\}$. □

Lemma 8.15. *Let G and H be connected graphs. Then each $(u,u')-(v,v')$ walk in $G\times H$ can be written as $W \times W'$ where W is a $u-v$ walk in G and W' is a $u'-v'$ walk in H.*

Proof. Let $X = (u,u'), (u,u')(w_2,w_2'), (w_2,w_2'), \ldots, (v,v')$ be a walk in $G\times H$. Then from the definition of adjacency in the tensor product we get that $W = u, uw_2, w_2, \ldots, w_{k-1}v, v$ is a walk in G and $W' = u', u'w_2', w_2', \ldots, w_{k-1}'v', v'$ is a walk in H. Furthermore it is clear that $X = W \times W'$. □

Proposition 8.16. *Let G and H be connected graphs. If there is no $u-v$ walk in G the same length as a $u'-v'$ walk in H, then $G \times H$ is disconnected and (u,v) belongs to a different component to (u',v').*

Proof. Assume the conditions of the proposition are satisfied. By Lemma 8.15 any $(u,u')-(v,v')$ walk Z in $G\times H$ can be written $Z = W \times W'$, where W is a $u-v$ walk in G and W' a $u'-v'$ walk in H. But W and W' must be the same length for this to be the case, which is a contradiction. And so there is no walk from (u,u') to (v,v') implying that $G\times H$ is disconnected and (u,u') and (v,v') belong to different components. □

Proposition 8.17. *Let G and H be connected graphs. If there is a $u-v$ walk in G the same length as a $u'-v'$ walk in H, then $d_{G\times H}((u,u'),(v,v')) = \min\{n \in \mathbb{N} \mid$ there exist $u-v$ and $u'-v'$ walks of length n in G and H respectively$\}$*

Proof. Let $k = \min\{n \in \mathbb{N} \mid$ there exists $u - v$ and $u' - v'$ walks of length n in G and H respectively$\}$ and let W and W' be $u - v$ and $u - v'$ walks of length k in G and H respectively. Then by Lemma 8.14 we have that $d_{G\times H}((u,u'),(v,v')) \leq k = \min\{n \in \mathbb{N} \mid$ there exist $u - v$ and $u' - v'$ walks of length n in G and H respectively$\}$.

Furthermore, each $(u,u') - (v,v')$ walk Z in $G \times H$, can be written $Z = W \times W'$ where W and W' are $u - v$ and $u' - v'$ walks of length t in G and H respectively. Hence $d_{G\times H}((u,u'),(v,v')) \geq k$. □

Proposition 8.18. *Let G and H be connected graphs, let g be a scaling function and let one of the following two conditions hold:*

1. *G and H are trees and g is subadditive,*
2. *g is linear.*

Then we have

$$\gamma_s^g(G \times H) \leq 2g^*(\frac{(2g(1)+1)}{2}\max\{\gamma_s^g(G), \gamma_s^g(H)\})$$

when $\mathrm{rad}(G) \neq \mathrm{rad}(H)$ *and*

$$\begin{aligned}\gamma_s^g(G \times H) \leq \min\{&g^*(\frac{2g(1)+1}{2}\gamma_s^g(G)) + g^*(\frac{2g(1)+1}{2}\gamma(G)_s^g + 1),\\ &g^*(\frac{2g(1)+1}{2}\gamma_s^g(H)) + g^*(\frac{2g(1)+1}{2}\gamma_s^g(H) + 1)\}\end{aligned}$$

when $\mathrm{rad}(G) = \mathrm{rad}(H)$.

Proof. Without loss of generality let $\mathrm{rad}(H) \leq \mathrm{rad}(G)$. Let $u \in V(G)$ be a vertex in the centre of G, $v \in V(H)$ a vertex in the centre of H and $v' \in V(H)$ a vertex adjacent to v.

Assume $\mathrm{rad}(H) < \mathrm{rad}(G)$. Then let h be the S-cast with $h(u,v) = h(u,v') = g^*(\mathrm{rad}(G))$ and $h(x,y) = 0$ otherwise. For each $x \in V(G)$ and $y \in V(H)$, we have $d_G(x,u) \leq \mathrm{rad}(G)$ and $d_H(y,v) < \mathrm{rad}(G)$. If $d_G(x,u)$ has the same parity as $d_H(y,v)$ and condition 1 holds we apply Proposition 8.17 and Lemma 8.5 (and if condition 2 holds we apply Proposition 8.17 and Corollary 8.6) to get $d_{G\times H}((x,y),(u,v)) \leq \max\{d_G(x,u), d_H(y,v)\} \leq \mathrm{rad}(G) \leq (g \circ h)(u,v)$. If $d_G(x,u)$ and $d_H(y,v)$ do not have the same parity, then $d_G(x,u)$ has the same parity as $d_H(x,v')$. Hence $d_{G\times H}((x,y),(u,v')) \leq \max\{d_G(x,u), d_H(y,v')\} \leq$

$\mathrm{rad}(G) \leq (g \circ h)(u, v')$ and thus h' is dominating. Consequently $\gamma_s^g(G \times H) \leq 2g^*(\mathrm{rad}(G)) \leq 2g^*(\frac{2g(1)+1}{2}\gamma(G)) \leq 2g^*(\frac{(2g(1)+1)}{2}\max\{\gamma_s^g(G), \gamma_s^g(H)\})$.

Assume $\mathrm{rad}(G) = \mathrm{rad}(H)$. Then let h be the S-cast with $h(v, u) = g^*(\mathrm{rad}(G))$, $h(u, v') = g * (\mathrm{rad}(G) + 1)$ and $h(x, y) = 0$ otherwise. For each $x \in V(G)$ and $y \in V(H)$, we have $d_G(x, u) \leq \mathrm{rad}(G)$ and $d_H(y, v) < \mathrm{rad}(G)$. If $d_G(x, u)$ has the same parity as $d_H(y, v)$ and condition 1 holds we apply Proposition 8.17 and Lemma 8.5 (and if condition 2 holds we apply Proposition 8.17 and Corollary 8.6) we have $d_{G\times H}((x, y), (u, v)) \leq \max\{d_G(x, u), d_H(y, v)\} \leq \mathrm{rad}(G) \leq (g \circ h)(u, v)$. If $d_G(x, u)$ and $d_H(y, v)$ do not have the same parity, then $d_G(x, u)$ has the same parity as $d_H(x, v')$. Hence $d_{G\times H}((x, y), (u, v')) \leq \max\{d_G(x, u), d_H(y, v')\} \leq \mathrm{rad}(G) + 1 \leq (g \circ h)(u, v')$ and thus h' is dominating. Consequently $\gamma_s^g(G \times H) \leq g^*(\mathrm{rad}(G)) + g^*(\mathrm{rad}(G)+1) \leq g^*(\frac{2g(1)+1}{2}\gamma(G)) + g^*(\frac{2g(1)+1}{2}\gamma(G)+1)$. A similar argument gives that $\gamma_s^g(G \times H) \leq g^*(\frac{2g(1)+1}{2}\gamma(H)) + g^*(\frac{2g(1)+1}{2}\gamma(H) + 1)$. Combining we get $\gamma_s^g(G \times H) \leq \min\{g^*(\frac{2g(1)+1}{2}\gamma(G)) + g^*(\frac{2g(1)+1}{2}\gamma(G) + 1), g^*(\frac{2g(1)+1}{2}\gamma(H)) + g^*(\frac{2g(1)+1}{2}\gamma(H) + 1)\}$. □

8.4 The Strong Product

Definition 8.19. *Let G and H be two graphs. Then the strong product of G and H, denoted $G \boxtimes H$, is the graph with vertex set $V(G \boxtimes H) = V(G) \times V(H)$ the Cartesian product of $V(G)$ and $V(H)$ and an edge set $E(G \boxtimes H) = E(G \square H) \cup E(G \times H)$.*

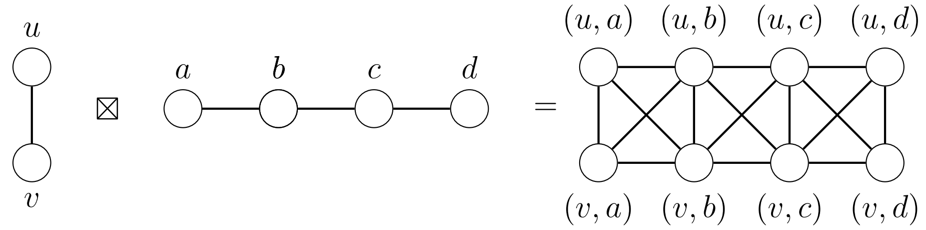

Figure 13: $P_2 \boxtimes P_4$

Proposition 8.20. *If G and H are connected graphs, then $G \boxtimes H$ is connected.*

Proof. $G \boxtimes H$ can be thought of as $G \square H$ with additional edges. Since $G \square H$ is connected, so too is $G \boxtimes H$. □

Proposition 8.21. *If G and H are connected graphs, $u, v \in V(G)$ and $u', v' \in V(H)$. Then $d_{G\boxtimes H}((u,u'),(v,v')) \leq \max\{d_G(u,v), d_H(u',v')\}$.*

Proof. Let $u, v \in V(G)$ and $u', v' \in V(H)$ with $d_G(u,v) = k$ and $d_H(u',v') = l$. Let $P = u, uw_2, w_2, \ldots, w_{k-1}v, v$ be a $u - v$ geodesic and $P = u', u'w_2', w_2', \ldots, w_{l-1}'v', v'$ a $u' - v'$ geodesic. Without loss of generality assume $k \leq l$ and let $l - k = d$. Then

$$(u,u'), (u,u')(w_2,w_2'), (w_2,w_2'), \ldots, (w_{k-1}, w_{l-d-1}')(v, w_{l-d}), (v, w_{l-d}),$$
$$(v, w_{l-d})(v, w_{l-d+1}), (v, w_{l-d+1}), \ldots, (v, w_{l-1})(v, v'), (v, v')$$

is a $(u,v) - (u',v')$ path of length $\max\{d_G(u,v), d_H(u',v')\}$. □

Corollary 8.22. *If G and H are connected graphs then* $\mathrm{rad}(G\boxtimes H) \leq \max\{\mathrm{rad}(G), \mathrm{rad}(H)\}$.

Proposition 8.23. *Let G and H be connected graphs, let g be a linear scaling function and suppose one of the following two conditions hold:*

1. *G and H are trees and g is subadditive,*
2. *g is linear.*

Then we get that

$$\gamma_s^g(G \boxtimes H) \leq g^*(\frac{2g(1)+1}{2} \max\{\gamma_s^g(G), \gamma_s^g(H)\}).$$

Proof. If condition 1 holds then by Corollary 8.22 and Lemma 8.5 (if condition 2 holds then by Corollary 8.22 and Lemma 8.5) we have that $\gamma_s^g(G \boxtimes H) \leq g^*(\mathrm{rad}(G \boxtimes H)) \leq g^*(\max\{\mathrm{rad}(G), \mathrm{rad}(H)\}) \leq g^*(\frac{2g(1)+1}{2} \max\{\gamma_s^g(G), \gamma_s^g(H)\})$. □

9 Γ-Theories

The results concerning the adjunction between scaling and cost functions are independent of the broadcast domination problem. Consequently it can be applied more generally to a number of other combinatorial problems.

Let G be a graph. Then $\overline{\mathbb{N}}^{V(G)}$ is the set of all S-casts on G. In the case of S-cast domination we restrict ourselves to the subset $S^g(G) \subseteq \overline{\mathbb{N}}^{V(G)}$ of dominating S-casts and the central problem concerns finding $\gamma_s^g(G) = \min\{\sum_{v \in V(G)} h(v) : h \in S^g(G)\}$. Alternatively we could consider the subset $C^{g^*}(G) \subseteq \overline{\mathbb{N}}^{V(G)}$ of dominating broadcasts with respect to the cost function g^* and we study $\gamma_c^{g^*}(G) = \min\{\sum_{v \in V(G)}(g^* \circ f)(v) \mid f \in C^{g^*}(G)\}$. As we have seen, minimising $C^{g^*}(G)$ is the same as minimising $S^g(G)$ and this useful relationship between these two sets often allows us to transport results from the one framework to the other.

To formalise precisely for which combinatorial problems these results hold, we introduce Γ-theories.

Definition 9.1. *Let S be the set of all scaling functions and T the collection of all graphs. Let $(C_{g,G})_{(g,G)\in S\times T}$ be a family of sets such that $C_{g,G} \subseteq \overline{\mathbb{N}}^{V(G)}$. Define $S_{g,G} = \{h \in \overline{\mathbb{N}}^{V(G)} \mid g \circ h \in C_{g,G}\}$. The set $C_{g,G}$ is all allowed broadcasts on (G, g) and $S_{g,G}$ is the set of allowed S-casts on (G, g). Similarly to Proposition 3.47 we have that composing with g and with g^* induces an adjunction between $S_{g,G}$ and $S_{g,G}$ and consequently a closure operator on $C_{g,G}$ and an interior operator on $S_{g,G}$. We call C a Γ-theory if, for each scaling function g and graph G, it is the case that $C_{g,G}$ is closed with respect to the closure operator mentioned in Proposition 3.49.*

Requiring that $C_{g,G}$ be closed is equivalent to requiring that if $f \in C_{g,G}$ then $g^* \circ f \in S_{g,G}$.

Example 9.2. *Let $C_{g,G}$ be the set of all dominating broadcasts on G. Then $S_{g,G}$ is the set of all functions h such that $g \circ h$ is a dominating broadcast. This is just the set of all dominating S-casts. By Lemma 3.46 we know that $f \leq g \circ g^* \circ f$ and consequently that $g \circ g^* \circ f$ is dominating and an element of $C_{g,G}$. Thus C is closed and consequently a Γ-theory.*

If $f \in C_{g,G}$ and $k \geq f$ implies $k \in C_{g,G}$, we say broadcasts in $C_{g,G}$ satisfy a *superhereditary* property. In this case, C is a Γ-theory as $g \circ g^* \circ f \geq f$. If instead $f \in C_{g,G}$ and $k \leq f$ implies $k \in C_{g,G}$ then we say broadcasts in $C_{g,G}$ satisfy a hereditary property.

Definition 9.3. *Let C be a Γ-theory. Then for each scaling function g and graph G we define*

1. $\gamma_C^{g^*}(G) = \min\{\sum_{v \in V}(g^* \circ f)(v) \mid f \in C_{g,G}\}$,
2. $\gamma_S^{g}(G) = \min\{\sum_{v \in V} h(v) \mid h \in S_{g,G}\}$.

Theorem 9.4. *Let C be a Γ-theory, g a scaling function and G a graph. Then $\gamma_C^{g^*}(G) = \gamma_S^{g}(G)$.*

Proof. Let $h \in S_{g,G}$ such that $\sum_{v \in V} h(v) = \gamma_S^g(G)$. The cost of $g \circ h$ is given by $\sum_{v \in V}(g^* \circ g \circ h)(v)$. But $g^* \circ g \circ h \leq h$ by Lemma 3.46 and thus $\gamma_C^{g^*}(G) \leq \gamma_S^g(G)$.

Let $f \in C_{g,G}$ be a broadcast such that $\sum_{v \in V}(g^* \circ f)(v) = \gamma_C^{g^*}(G)$ and let $g^* \circ f = h$. Then $w_s^g(h) = \sum_{v \in V}(g^* \circ f)(v) = \gamma_C^{g^*}(G)$ and thus $\gamma_S^g(G) \leq \gamma_C^{g^*}(G)$. $\square$

In order to handle hereditary properties we must make some minor adjustments. To illustrate this point we look at how we might generalise broadcast independence to the S-cast setting.

The notion of an independent S-cast is what we might expect.

Definition 9.5. *Given a graph G, an independent broadcast f is a broadcast where $v \notin N_f(u)$ for any $u, v \in V_f^+$.*

Definition 9.6. *Let G be a graph and g a scaling function.*

1. *An independent S-cast h is an S-cast where $g \circ h$ is an independent broadcast.*
2. *A maximal independent S-cast is an independent S-cast h with the property that any S-cast k with $k > h$ is not an independent S-cast.*

If we were to consider the set of independent broadcasts $C_{g,G}$ on G and further the set of independent S-casts $S_{g,G}$, we find that $C_{g,G}$ is not closed with respect to the usual adjunction between $C_{g,G}$ and $S_{g,G}$.

Example 9.7. *Let g be a scaling function with $g(1) = 2$ and consider the independent broadcast f on P_3 where $f(u) = f(v) = 1$, u and v are the end vertices and $f(w) = 0$ where w is the middle vertex. Then $g^* \circ f$ is not independent as $g \circ g^* \circ f(u) = g \circ g^* \circ f(v) = 2 \neq 0$ and so $v \in N_{g \circ g^* \circ f}(u)$.*

If we consider the intuitive definition of a left adjoint the problem becomes apparent, as $g^*(n)$ is defined as the least number making $g(g^*(n))$ greater than or equal to n. In the case where $g(g^*(n))$ is strictly greater than n we can lose independence.

This can be fixed by considering the right adjoint g_* of the scaling function g. Here $g_*(n)$ is defined as the largest value making $g(g_*(n))$ less than or equal to n.

As it stands the right adjoint of a scaling function need not exist at all. Interchanging the definitions of cost and scaling functions solves this problem, namely requiring for a scaling function g that $g(\infty) = \sup\{g(x) \mid x \in \mathbb{N}\}$ and for a cost function k that $k(\infty) = \infty$. This follows from Theorem 3.35 and Theorem 3.37.

If we want both the left and right adjoints of a scaling function to exist then we must require that $\infty = g(\infty) = \sup\{g(x) \mid x \in \mathbb{N}\}$, which is the same as requiring that $\infty = \sup\{g(x) \mid x \in \mathbb{N}\}$.

Proposition 9.8. *The scaling function g has both left and right adjoints if and only if g is unbounded on $\mathbb{N}$.*

Proof. Assume left and right adjoints exist for g. Then $\infty = \sup\{g(x) \mid x \in \mathbb{N}\}$. If $g(x) \leq k < \infty$ for each $x \in \mathbb{N}$ then $\sup\{g(x) \mid x \in \mathbb{N}\} \leq k < \infty$ which is a contradiction.

If g is unbounded then, because g is nondecreasing, $g(\infty) = \infty$. Thus a left adjoint exists. Furthermore $\sup\{g(x) \mid x \in \mathbb{N}\} = \infty = g(\infty)$ by definition. Thus a right adjoint exists. □

For the remainder of this section we will work with scaling functions in which the right adjoint exists. This right adjoint will be the cost function. And just as in Proposition 3.47 we get an induced adjunction between the set of broadcasts and S-casts. The main difference now is that we have an interior operator on the set of broadcasts and a closure operator on the set of S-casts.

We can now define a co-Γ-theory.

Definition 9.9. *Let S be the set of all scaling functions (with right adjoints) and T the collection of all graphs. Let $(C_{g,G})_{(g,G)\in S\times T}$ be a family of sets such that $C_{g,G} \subseteq \overline{\mathbb{N}}^{V(G)}$. Define $S_{g,G} = \{h \in \overline{\mathbb{N}}^{V(G)} : g \circ h \in C_{g,G}\}$. The set $C_{g,G}$ is all allowed broadcasts on (G, g) and $S_{g,G}$ is the set of allowed S-casts on (G, g). Similarly to Proposition 3.47 we have that composing with g and with g^* induces an adjunction between $S_{g,G}$ and $S_{g,G}$ and consequently an interior operator on $C_{g,G}$ and a closure operator on $S_{g,G}$. We call C a co-Γ-theory if, for each scaling function g and graph G, it is the case that $C_{g,G}$ is closed with respect to the interior operator mentioned in Proposition 3.49.*

With co-Γ-theories we want to maximise the value of a broadcast.

Definition 9.10. *Let C be a co-Γ-theory. Then for each scaling function g and graph G let*

1. $i_C^{g_*}(G) = \max\{\sum_{v\in V}(g_* \circ f)(v) \mid f \in C_{g,G}\}$,
2. $i_S^{g}(G) = \max\{\sum_{v\in V} h(v) \mid h \in S_{g,G}\}$.

Example 9.11. *Let $C_{g,G}$ be the set of all independent broadcasts on G. Then $S_{g,G}$ is the set of all functions h such that $g \circ h$ is an independent broadcast. This is just the set of all independent S-casts. By Lemma 3.46 we know that $f \geq g \circ g^* \circ f$ and consequently that $g \circ g^* \circ f$ is independent and an element of $C_{g,G}$. Thus C is closed and consequently a co-Γ-theory.*

Example 9.12. *A broadcast is a packing broadcast when each vertex hears at most one broadcasting vertex. Given a scaling function g we say an S-cast h is a packing S-cast if $g \circ h$ is a packing broadcast.*

If we let $C_{g,G}$ be all packing broadcasts then by an argument similar to that above we get that $S_{g,G}$ is the set of all packing S-casts. Furthermore if f is a packing and $f' \leq f$ then f' is a packing (packing is a hereditary property) and so $C_{g,G}$ is closed with respect to the interior operator and thus a co-Γ-theory.

If $C_{g,G}$ is the set of all broadcasts satisfying a hereditary property then C is a co-Γ-theory.

Co-Γ-theories also satisfy a theorem relating $i_C^{g_*}(G)$ and $i_S^{g}(G)$.

Theorem 9.13. *Let C be a co-Γ-theory, g a scaling function and G a graph. Then $i_C^{g_*}(G) = i_S^g(G)$.*

Proof. Let $h \in S_{g,G}$ such that $w_s^g(h) = i_S^g(G)$. Then $w_s^g(g \circ h) = w_s^g(g_* \circ g \circ h)$ and $g_* \circ g \circ h \geq h$ by Lemma 3.46 and so we conclude that $i_S^g(G) \leq i_C^{g_*}$.

Let $f \in C_{g,G}$, then the cost of f is equal to the cost of the S-cast $g_* \circ f$. This gives us that $i_C^{g_*} \leq i_S^g$ which completes the proof. □

So for any Γ-theory or co-Γ-theory the parameters for cost and scaling functions coincide. Consequently one only needs to solve either of the two computational problems and there will be a solution that immediately carries over to the other.

10 Concluding Remarks

In this thesis we have found a number of bounds and results pertaining to specific classes of scaling functions, the main ones being subadditive scaling function, linear scaling functions and superadditive scaling functions. Here we summarise the results which hold for scaling functions belonging to each class.

g is a superadditive scaling function For any graph G:

- $S^g(G) \in \mathcal{P}$.
- There always exists an efficient optimal dominating S-cast on (G, g).
- There always exists an independent optimal dominating S-cast on (G, g).
- There exists a spanning tree T of G, such that $\gamma_s^g(T) = \gamma_s^g(G)$.

g is a subadditive scaling function

- Let P_n be the path of n vertices. Then $\gamma_s^g(P_n) = \lceil \frac{n}{2g(1)+1} \rceil$.
- Let T be a tree. Then $\gamma_s^g(T) \geq \lceil \frac{2\,\mathrm{rad}(T)}{2g(1)+1} \rceil$.
- Let S and T be trees. Then $\gamma(S \,\square\, T) \leq g^*(\frac{2g(1)+1}{2}(\gamma_s^g(S) + \gamma_s^g(T)))$.
- Let S and T be trees. Then $\gamma_s^g(S \boxtimes T) \leq g^*(\frac{2g(1)+1}{2} \max\{\gamma_s^g(S), \gamma_s^g(T)\})$.
- Let S and T be trees. Then

  $$\gamma_s^g(S \times T) \leq 2g^*(\frac{(2g(1)+1)}{2} \max\{\gamma_s^g(S), \gamma_s^g(T)\})$$

  when $\mathrm{rad}(S) \neq \mathrm{rad}(T)$ and

  $$\begin{aligned} \gamma_s^g(S \times T) \leq \min\{ & g^*(\frac{2g(1)+1}{2}\gamma(S)) + g^*(\frac{2g(1)+1}{2}\gamma(S)+1), \\ & g^*(\frac{2g(1)+1}{2}\gamma(T)) + g^*(\frac{2g(1)+1}{2}\gamma(T)+1)\} \end{aligned}$$

  when $\mathrm{rad}(T) = \mathrm{rad}(S)$.

g **is a linear scaling function** For any graphs G and H:

- All of the above results.
- $\gamma(G \,\square\, H) \leq g^*(\frac{2g(1)+1}{2}(\gamma_s^g(G) + \gamma_s^g(H)))$.
- $\gamma_s^g(G \boxtimes H) \leq g^*(\frac{2g(1)+1}{2}\max\{\gamma_s^g(G), \gamma_s^g(H)\})$.
- $$\gamma_s^g(G \times H) \leq 2g^*(\frac{(2g(1)+1)}{2}\max\{\gamma_s^g(G), \gamma_s^g(H)\})$$

  when $\text{rad}(G) \neq \text{rad}(H)$, and

  $$\begin{aligned}\gamma_s^g(G \times H) \leq \min\{&g^*(\frac{2g(1)+1}{2}\gamma(G)) + g^*(\frac{2g(1)+1}{2}\gamma(G)+1),\\ &g^*(\frac{2g(1)+1}{2}\gamma(H)) + g^*(\frac{2g(1)+1}{2}\gamma(H)+1)\}\end{aligned}$$

  when $\text{rad}(G) = \text{rad}(H)$.

g **is a strictly superadditive scaling function**

- Let G be a graph. Then every optimal dominating S-cast on (G, g) is radial.

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